Modal Characterisation of Simulation Relations in Probabilistic Concurrent Games

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Abstract

Probabilistic game structures combine both nondeterminism and stochasticity, where players repeatedly take actions simultaneously to move to the next state of the concurrent game. Probabilistic alternating simulation and bisimulation are important tools to compare the behaviour of different probabilistic game structures. In this paper, we present a sound and complete modal characterisation of these two relations by proposing a new modal logic based on probability distributions. This logic enables a player to enforce a property in the next state or distribution. We further extend the logic with fixpoint operators, which also characterises the simulation relations. This logic can express a lot of interesting properties in practical applications.

Keywords: Concurrent games, probabilistic alternating simulation, probabilistic alternating bisimulation, modal logic, logic characterisation

1. Introduction

Simulation relations and bisimulation relations [1] are important research topics in concurrency theory. In the classical model of labelled transition systems (LTS), simulation and bisimulation have been proved useful for compar-

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ing the behaviour of concurrent systems, with many applications, for example, to verifying communication protocols. The modal characterisation problem has been studied both in classical and in probabilistic systems, i.e., the Hennessy-Milner logic (HML) [2] that characterises image-finite LTS, and various modal logics have been proposed to characterise strong and weak probabilistic (bi)simulation in the model of probabilistic automata [3, 4, 5]. To study multi-player games, the concurrent game structure (GS) [6] is a model that defines a system that evolves while interacting with outside *players*. As a player's behaviour is not fully specified within a system, GS are often also known as *open* systems. Alternating simulation (A-simulation) is defined in GS focusing on players' ability to enforce temporal properties specified in alternating-time temporal logic (ATL) [6], and A-simulation is shown to be sound and complete for a fragment of ATL [7].

In this paper, we work on the model of probabilistic game structure (PGS) which has probabilistic transitions. PGS also allows probabilistic (or mixed) choices of players. This makes PGS an appealing model for concurrent games with application domains such as robotics, security, and autonomous transport. The simulation relation in PGS, called probabilistic alternating simulation (PA-simulation), has been shown to preserve a fragment of probabilistic alternating-time temporal logic (PATL) under *mixed strategies* [8]. Given the classical results of modal characterisations for (non-probabilistic) LTS, probabilistic automata, as well as for (non-probabilistic) game structures, we investigate if a similar correspondence exists for processes and modal logics in the domain of concurrent games with probabilistic transitions and mixed strategies. We find that such a correspondence still holds by adapting a modal logic with nondeterministic distributions extended from the work of [4].

Contributions. This paper studies modal characterisation of the probabilistic alternating simulation relation and the probabilistic alternating bisimulation relation in probabilistic game structures. For our research objective, we propose a novel modal logic, $\mathcal{L}^{\oplus}$, based on probability distributions. This new logic expresses a player's power to enforce a property in the next state or distribution. The logic also incorporates both probabilistic and nondeterminstic features that need to be considered during the two-player interplay. We prove that $\mathcal{L}^{\oplus}$ characterises probabilistic alternating bisimulation (PA-bisimulation), while its sub-logic $\mathcal{L}^{\ominus}$ characterises probabilistic alternating simulation (PA-simulation). Another contribution of this paper is the introduction of a fixpoint logic, which also characterises the PA-simulation and PA-bisimulation relations, extended from the modal logic $\mathcal{L}^{\oplus}$ with fixpoint operators. The expressive power of this fixpoint logic has been discussed and illustrated by several examples.

Note that this paper extends on a conference paper presented at TASE'20 [9], which was an extended abstract with many detailed proofs omitted. The modal logic presented in the conference paper [9] is actually $\mathcal{L}^{\ominus}$, a fragment of the new modal logic $\mathcal{L}^{\oplus}$ proposed in this paper. Moreover, the additional contribution of this paper consists of the sound and complete modal characterisation of the probabilistic alternating bisimulation relation by $\mathcal{L}^{\oplus}$.

Structure of the paper. We present some necessary preliminaries in Section 2, and give the formal definitions of probabilistic alternating simulation and bisimulation in Section 3. A new modal logic with nondeterministic distributions, $\mathcal{L}^{\oplus}$, is introduced in Section 4. We continue to show that its sublogic $\mathcal{L}^{\ominus}$ has a correct characterisation of probabilistic alternating simulation (Section 5), while $\mathcal{L}^{\oplus}$ itself characterises probabilistic alternating bisimulation (Section 6) for probabilistic concurrent games. In Section 7, the logic $\mathcal{L}^{\oplus}$ is then extended with variables and fixpoint operators, resulting in a probabilistic alternating-time μ-calculus (PAMu), and the logic characterisation of probabilistic alternating bisimulation (and simulation) can be extended to PAMu as well. We discuss related work in Section 8 and conclude the paper with possible future work in Section 9.

2. Preliminaries

A *discrete probability distribution* Δ over a set S is a function of type $S \to [0,1]$, where $\sum_{s \in S} \Delta(s) = 1$. We write $\mathcal{D}(S)$ for the set of all such distributions, ranged over by symbols $\Delta, \Theta, \ldots$. Given a set $T \subseteq S$, $\Delta(T) = \sum_{s \in T} \Delta(s)$, i.e., the probability for the given set T. Given an index set I, a list of distributions $\langle \Delta_i \rangle_{i \in I}$ and a list of values $\langle p_i \rangle_{i \in I}$ where $p_i \in [0,1]$ for all $i \in I$ and $\sum_{i \in I} p_i = 1$, we have that $\sum_{i \in I} p_i \Delta_i$ is also a distribution, i.e., the weighted sum of distributions is again a distribution. If $|I| = 2$ we may also write $\Delta_1 \oplus_\alpha \Delta_2$ for the distribution $p_1 \Delta_1 + p_2 \Delta_2$ where $p_1 = \alpha$ and $p_2 = 1 - \alpha$. For $s \in S$, $s_\star$ represents a *point (or Dirac) distribution* satisfying $s_\star(s) = 1$ and $s_\star(t) = 0$ for all $t \neq s$. Given $\Delta \in \mathcal{D}(S)$, we define $\lceil \Delta \rceil$ as the set $\{s \in S \mid \Delta(s) > 0\}$, which is the *support* of Δ.

To this point, we present the model of (non-probabilistic) game structure for two players $\mathcal{I}$ and $\mathcal{II}$, though we believe that the results in this paper for two players can be extended to handle a finite set of players as in the standard concurrent game structures [6]. Each player has complete information about the PGS at any time during a play. Let `Prop` be a finite set of propositions.

Definition 1. *A game structure (GS) is a tuple $\langle S, s_0, L, \mathtt{Act}, \delta \rangle$, where*

- *S is a finite set of states, with s_0 the initial state;*
- *$L : S \to 2^{\mathit{Prop}}$ is the labelling function which assigns to each state $s \in S$ a set of propositions true in s;*
- *$\mathtt{Act} = \mathtt{Act}_{\mathcal{I}} \times \mathtt{Act}_{\mathcal{II}}$ is a finite set of joint actions, where $\mathtt{Act}_{\mathcal{I}}$ and $\mathtt{Act}_{\mathcal{II}}$ are, respectively, the sets of actions for players $\mathcal{I}$ and $\mathcal{II}$;*
- *$\delta : S \times \mathtt{Act} \to S$ is the transition function.*

Given the deterministic transition of a GS, a game play is a sequence of the form $s_0 \xrightarrow{\langle a_1, b_1 \rangle} s_1 \xrightarrow{\langle a_2, b_2 \rangle} s_2 \ldots \xrightarrow{\langle a_k, b_k \rangle} s_k$, where $s_i = \delta(s_{i-1}, \langle a_i, b_i \rangle)$ for all $1 \leq i \leq k$. During the game, the players choose their next moves *simultaneously* for each step, so that this style of game play is often known

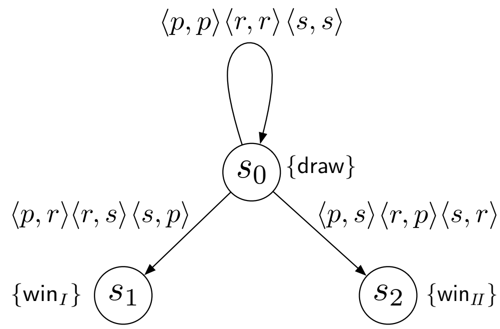

Figure 1: The PGS for the repeated rock-paper-scissors game.

as concurrent games [10]. In the above play, in each state s_i, one may think of the players making their best efforts to win this game. Let us consider the following example.

Example 1. *Figure 1 presents the GS of two players repeatedly playing the rock-paper-scissors game.*[1] *It has three states s_0, s_1, and s_2, with s_0 being the initial state. Each state is labelled with an atomic proposition indicating the result of the current round (indicating which player wins or there is a draw). For instance, in state s_1 player $\mathcal{I}$ wins the game. Actions of the players are r (representing "rock"), p (representing "paper"), and s (representing "scissors"). The joint actions $\langle a_1, a_2\rangle$ with $a_1, a_2 \in \{r, p, s\}$ are depicted along with the transitions. The function δ describes the transition function as shown in Figure 1. The winning states s_1 and s_2 are absorbing, i.e., all actions from there make self-transitions, and the game effectively terminates there.*

Informally, it is sometimes impossible for a player, say player $\mathcal{I}$, to find a deterministic strategy that picks an action to maximize her chance to win the game. Taking Example 1, in state s_0, whichever (deterministic) action a player $\mathcal{I}$ chooses, there exists a (counter) action b from player $\mathcal{II}$, such that $\langle a, b\rangle$

[1] A similar example was used in [11], where a concurrent stochastic structure (CSG), as defined in [11], is a game structure (GS) and also a PGS to be defined in the following, in the sense that all the probability distributions involved in the CSG are point distributions.

leads to a losing state for player $\mathcal{I}$. Consequently, following the classic results on matrix games [12], we allow a player to have probabilistic (or randomized) behaviour, which is defined in the following.

Definition 2. *A* mixed action *of player i is a function from states to distributions on $\mathtt{Act}_i$, ranged over by $\pi, \pi_1, \sigma \ldots$. We write Π_i for the set of mixed actions from player i. In particular, given $a \in \mathtt{Act}_i$, write $a_\star$ for the* deterministic *mixed action for player i which always chooses action a with probability* 1 *in all states.*

To show the effectiveness of probabilistic choices, in the rock-paper-scissors game in Example 1 (see Figure 1), for both players, the mixed action with probability $\frac{1}{3}$ for each of the actions (r, p and s) is known as the *optimal* strategy for both players, in the sense that it guarantees the chance to eventually win the game with probability at least $\frac{1}{2}$.

Besides player strategies, randomization can also be used for modelling uncertainty in system behaviour. For example, security market can be regarded as a game between players, or between a player and the market, plus direct or indirect influences from a variety of unknown factors [11]. In the following, we extend the game structure (GS) model to allow probabilistic transitions.

Definition 3. *A probabilistic game structure (PGS) is a tuple $\langle S, s_0, L, \mathtt{Act}, \delta\rangle$, where*

- *S is a finite set of states, with s_0 the initial state;*
- *$L : S \to 2^{\mathit{Prop}}$ is the labelling function;*
- *$\mathtt{Act} = \mathtt{Act}_{\mathcal{I}} \times \mathtt{Act}_{\mathcal{II}}$ is a finite set of joint actions;*
- *$\delta : S \times \mathtt{Act} \to \mathcal{D}(S)$ is a* probabilistic *transition function.*

The new probabilistic transition function δ essentially captures the probabilistic aspect of the game structure — after performing an action in $\mathtt{Act}$, the game structure moves to a distribution over the states, instead of moving into one state. If in state s player $\mathcal{I}$ performs action a_1 and player $\mathcal{II}$ performs action a_2, then $\delta(s, \langle a_1, a_2 \rangle)$ is the distribution for the next states.

To this point, we generalise the transition function δ by $\widetilde{\delta}$ to handle mixed actions. (Note that the domain of δ is finite as both S and $\mathtt{Act}$ are finite, though the domain of $\widetilde{\delta}$ is infinite.) Given $\pi_1 \in \Pi_{\mathcal{I}}$ and $\pi_2 \in \Pi_{\mathcal{II}}$, for all $s, t \in S$, we define $\widetilde{\delta}(s, \langle \pi_1, \pi_2 \rangle)$ for the distribution that is reached by applying mixed actions π_1 and π_2 on state s.

$$\widetilde{\delta}(s, \langle \pi_1, \pi_2 \rangle)(t) = \sum_{a_1 \in \mathtt{Act}_{\mathcal{I}}, a_2 \in \mathtt{Act}_{\mathcal{II}}} \pi_1(s)(a_1) \cdot \pi_2(s)(a_2) \cdot \delta(s, \langle a_1, a_2 \rangle)(t).$$

In this definition, $\pi_1(s)(a_1)$ and $\pi_2(s)(a_2)$ are respectively the weights for actions a_1 and a_2, which contribute to the probability of a state, say t, to be reached by the transition from s.

Example 2. *We refer to the simple three-state PGS depicted in Figure 2, where the state space is* $\{s_0, s_1, s_2\}$, $\mathtt{Act}_{\mathcal{I}} = \{a\}$, *and* $\mathtt{Act}_{\mathcal{II}} = \{b_1, b_2\}$. *Let* $\epsilon > 0$ *be a tiny positive real number (say,* 10^{-10}*).*

- $\delta(s_0, \langle a, b_1 \rangle) = \Delta_1$, *where* $\Delta_1(s_1) = \Delta_1(s_2) = \frac{1}{2}$.

- $\delta(s_0, \langle a, b_2 \rangle) = \Delta_1'$, *where* $\Delta_1'(s_1) = \frac{1}{2} + \epsilon$ *and* $\Delta_1'(s_2) = \frac{1}{2} - \epsilon$.

One may observe that from s_0, *player* $\mathcal{II}$ *is only allowed to tweak the output distribution by a tiny bit.*

A convention for PGS figures. To improve readability of examples, we make the following convention for the PGS figures throughout this paper. If from a state there is no outgoing arrow, then that state is *absorbing*, as all actions from there make only self-transitions.

In this example, if player $\mathcal{II}$ performs b_1, then the resulting distribution will have equal weights on the two states s_1 and s_2. If player $\mathcal{II}$ performs b_2, the resulting distribution will have slightly more weight (i.e., by $2 \cdot \epsilon$) on s_1 than s_2. In fact, player $\mathcal{II}$ may also perform a mixed action σ satisfying $\sigma(s_0, b_1) = \frac{1}{2}$ and $\sigma(s_0, b_2) = \frac{1}{2}$. Informally, before her action, player $\mathcal{II}$ flips a fair coin, and

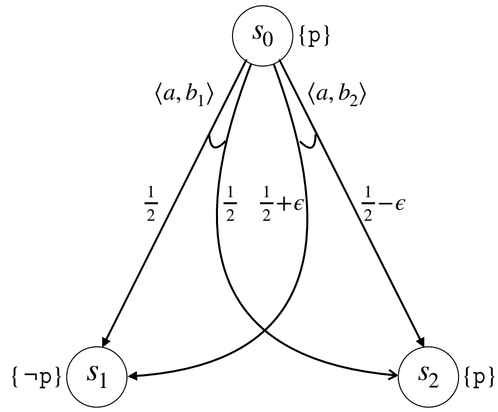

Figure 2: A probabilistic game structure $\mathcal{G}_1$

performs b_1 if the coin turns up heads and performs b_2 otherwise. In this case, we apply the generalised transition function, by $\widetilde{\delta}(s_0, \langle a_\star, \sigma\rangle) = \Delta$, where

$$\begin{aligned}\Delta(s_1) &= a_\star(s_0, a) \cdot \sigma(s_0, b_1) \cdot \delta(s_0, \langle a, b_1\rangle)(s_1) + a_\star(s_0, a) \cdot \sigma(s_0, b_2) \cdot \delta(s_0, \langle a, b_2\rangle)(s_1) \\ &= 1 \cdot \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \cdot \left(\frac{1}{2} + \epsilon\right) \\ &= \frac{1+\epsilon}{2}\end{aligned}$$

Similarly, $\Delta(s_2) = a_\star(s_0, a) \cdot \sigma(s_0, b_1) \cdot \delta(s_0, \langle a, b_1\rangle)(s_2) + a_\star(s_0, a) \cdot \sigma(s_0, b_2) \cdot \delta(s_0, \langle a, b_2\rangle)(s_2)$, which gives $\frac{1-\epsilon}{2}$.

Let $\leq \subseteq S \times S$ be a partial order, define $\leq_{Sm} \subseteq \mathcal{P}(S) \times \mathcal{P}(S)$, by $P \leq_{Sm} Q$ if for all $t \in Q$ there exists $s \in P$ such that $s \leq t$. In the literature this definition is known as the 'Smyth order' [13, 14] regarding '$\leq$'.

Relations in probabilistic systems usually require a notion of *lifting* [15], which extends the relations to the domain of distributions.[2]

Definition 4. *Let S, T be two sets and $\mathcal{R} \subseteq S \times T$ be a relation, then $\overline{\mathcal{R}} \subseteq \mathcal{D}(S) \times \mathcal{D}(T)$ is a* lifted relation *defined by $\Delta\,\overline{\mathcal{R}}\,\Theta$ if there exists a* weight *function $w : S \times T \to [0, 1]$ such that*

[2]In a probabilistic system without explicit user interactions, state s is simulated by state t if for every $s \xrightarrow{a} \Delta_1$ there exists $t \xrightarrow{a} \Delta_2$ such that Δ_1 is simulated by Δ_2.

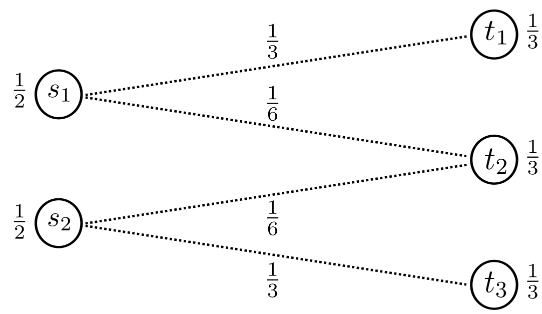

Figure 3: An example showing how to lift one relation.

- $\sum_{t \in T} w(s,t) = \Delta(s)$ *for all* $s \in S$,
- $\sum_{s \in S} w(s,t) = \Theta(t)$ *for all* $t \in T$,
- $s \mathcal{R} t$ *for all* $s \in S$ *and* $t \in T$ *with* $w(s,t) > 0$.

The intuition behind the lifting is that each state in the support of one distribution may correspond to a number of states in the support of the other distribution, and vice versa. In the following section, we extend the notion of alternating simulation [7] to a probabilistic setting in the way of lifting. The next example is taken from [16] which shows how to lift a relation.

Example 3. *In Figure 3, we have two sets of states defined by* $S = \{s_1, s_2\}$ *and* $T = \{t_1, t_2, t_3\}$, *and a relation* $\mathcal{R} = \{(s_1,t_1), (s_1,t_2), (s_2,t_2), (s_2,t_3)\}$. *Suppose* $\Delta(s_1) = \Delta(s_2) = \frac{1}{2}$ *and* $\Theta(t_1) = \Theta(t_2) = \Theta(t_3) = \frac{1}{3}$, *we may establish* $\Delta \overline{\mathcal{R}} \Theta$. *To check this, we define a weight function* w *by setting* $w(s_1,t_1) = \frac{1}{3}$, $w(s_1,t_2) = w(s_2,t_2) = \frac{1}{6}$, *and* $w(s_2,t_3) = \frac{1}{3}$. *The dotted lines in the graph indicate the allocation of weights that is required to relate* Δ *to* Θ *via* $\overline{\mathcal{R}}$.

We present some properties of lifted relations. First of all, we show that, by combining pairs of distributions that are lift-related with the same probability on both sides, we get the resulting (combined) distributions lift-related.

Lemma 1. *Let* $\mathcal{R} \subseteq S \times T$ *and* $\langle p_i \rangle_{i \in I}$ *be an index satisfying* $\sum_{i \in I} p_i = 1$, *and* $\Delta_i \overline{\mathcal{R}} \Theta_i$ *for* $\Delta_i \in \mathcal{D}(S)$ *and* $\Theta_i \in \mathcal{D}(T)$ *for all* i, *then* $\sum_{i \in I} p_i \Delta_i \overline{\mathcal{R}} \sum_{i \in I} p_i \Theta_i$.

Proof: For all $i \in I$, suppose w_i is the weight function that establishes $\Delta_i \overline{\mathcal{R}} \Theta_i$. We show that the new weight function w defined by $w(s,t) = \sum_{i\in I}(p_i \cdot w_i(s,t))$ is the weight function that establishes $\sum_{i\in I} p_i\Delta_i \overline{\mathcal{R}} \sum_{i\in I} p_i\Theta_i$.

- For all $s \in S$, we have $\sum_{t\in T} w(s,t) = \sum_{t\in T}\sum_{i\in I} p_i \cdot w_i(s,t) = \sum_{i\in I} p_i \cdot \sum_{t\in T} w_i(s,t)$. Since w_i is the weight function that establishes $\Delta_i \overline{\mathcal{R}} \Theta_i$ for all i, we have $\sum_{t\in T} w_i(s,t) = \Delta_i(s)$. Therefore, $\sum_{t\in T} w(s,t) = \sum_{i\in I} p_i\Delta_i(s)$, as required.

- For all $t \in T$, we are able to show $\sum_{s\in S} w(s,t) = \sum_{i\in I} p_i\Theta_i(t)$, which is symmetric to the above case.

- Suppose $w(s,t) > 0$. Since $w(s,t) = \sum_{i\in I} p_i \cdot w_i(s,t)$, we have $w_i(s,t) > 0$ for some i. Therefore $s \mathcal{R} t$.

$\square$

The following lemma states that, given two related distributions, if we "split" a distribution on one side of the relation by an index set, then there exists a split on the other side by the same index set, so that the corresponding (sub)distributions with the same index are related by the lifted relation.

Lemma 2. *Let $\Delta \in \mathcal{D}(S)$, $\Theta \in \mathcal{D}(T)$, $\mathcal{R} \subseteq S \times T$, $\langle p_i\rangle_{i\in I}$ be a list of positive real values satisfying $\sum_{i\in I} p_i = 1$. If $\Delta \overline{\mathcal{R}} \Theta$, then*

1. *for all lists of distributions $\langle\Delta_i\rangle_{i\in I}$ with $\Delta_i \in \mathcal{D}(S)$ for all $i \in I$, satisfying $\Delta = \sum_{i\in I} p_i\Delta_i$, there exist $\langle\Theta_i\rangle_{i\in I}$ with $\Theta_i \in \mathcal{D}(T)$ such that $\Theta = \sum_{i\in I} p_i\Theta_i$ and $\Delta_i \overline{\mathcal{R}} \Theta_i$ for all $i \in I$;*
2. *for all lists of distributions $\langle\Theta_i\rangle_{i\in I}$ with $\Theta_i \in \mathcal{D}(T)$ for all $i \in I$, satisfying $\Theta = \sum_{i\in I} p_i\Theta_i$, there exist $\langle\Delta_i\rangle_{i\in I}$ with $\Delta_i \in \mathcal{D}(S)$ such that $\Delta = \sum_{i\in I} p_i\Delta_i$, and $\Delta_i \overline{\mathcal{R}} \Theta_i$ for all $i \in I$.*

Proof: It suffices to only prove the second part, as the first part is symmetric. Suppose $\Theta = \sum_{i\in I} p_i \cdot \Theta_i$, and w is the weight function that establishes $\Delta \overline{\mathcal{R}} \Theta$, we define Δ_i for each $i \in I$ by $\Delta_i(s) = \sum_{t\in T} w(s,t) \cdot \frac{\Theta_i(t)}{\Theta(t)}$ for all $s \in S$.

We first check that $\sum_{i\in I} p_i \cdot \Delta_i(s) = \Delta(s)$ for all $s \in S$, i.e., $\Delta = \sum_{i\in I} p_i \cdot \Delta_i$. Let $s \in S$, then $\sum_{i\in I} p_i \cdot \Delta_i(s) = \sum_{i\in I} p_i \sum_{t\in T} w(s,t) \cdot \frac{\Theta_i(t)}{\Theta(t)} = \sum_{t\in T} \frac{w(s,t)}{\Theta(t)} \cdot \sum_{i\in I} p_i \Theta(t)$. Since $\sum_{i\in I} p_i \Theta_i(t) = \Theta(t)$ by definition, we have $\sum_{i\in I} p_i \cdot \Delta_i(s) = \sum_{t\in T} w(s,t) = \Delta(s)$.

Next we show that $\Delta_i \overline{\mathcal{R}} \Theta_i$ for each $i \in I$. Define a weight function $w_i : S \times T \to [0,1]$ as follows. For all $s \in S$ and $t \in T$, $w_i(s,t) = w(s,t) \cdot \frac{\Theta_i(t)}{\Theta(t)}$. Then we verify the following three conditions.

1. $w_i(s,t) > 0$ implies $w(s,t) > 0$, therefore $s\,\mathcal{R}\,t$.
2. For all $t \in T$, we have $\sum_{s\in S} w_i(s,t) = \sum_{s\in S} w(s,t) \cdot \frac{\Theta_i(t)}{\Theta(t)} = \frac{\Theta_i(t)}{\Theta(t)} \cdot \sum_{s\in S} w(s,t) = \frac{\Theta_i(t)}{\Theta(t)} \cdot \Theta(t) = \Theta_i(t)$.
3. This case is similar to the above case. For all $s \in S$, we have $\sum_{t\in T} w_i(s,t) = \sum_{t\in T} w(s,t) \cdot \frac{\Theta_i(t)}{\Theta(t)} = \Delta_i(s)$.

□

3. Probabilistic Alternating Simulation Relations

In concurrency models, simulation and bisimulation relations are used to relate states with respect to their behaviours. For example, in a labelled transition system (LTS) $\langle S, A, \rightarrow \rangle$, where S is a set of states, A is a set of actions and $\rightarrow \subseteq S \times A \times S$ is the transition relation, we say state s is simulated by state t, written $s \leq t$, if for every $s \xrightarrow{a} s'$ there exists $t \xrightarrow{a} t'$ such that $s' \leq t'$. In this coinductive definition, state t is able to simulate state s by performing the same action a, with their destination states still related. Simulation is a useful tool in abstraction and refinement based verification, as intuitively, in the above case, t contains at least as much "behaviour" as s does. Bisimulation is a stronger relation which requires that the two related states have exactly the same pattern of behaviours.

In a two-player non-probabilistic game structure (GS), alternating simulation (A-simulation) is used to describe a player's ability to enforce certain temporal requirements regardless of the other player's behaviours [7]. In this paper we focus on the ability of player $\mathcal{I}$ in a two-player game. Since in a game structure a transition requires the participation of both parties, fixing player $\mathcal{I}$'s input leaves a set of possible next states depending on player $\mathcal{II}$'s inputs. An A-simulation $\leq^A \subseteq S \times S$ is defined in the model of GS (given in Definition 1) as follows. Let $s, t \in S$, s is A-simulated by t, written $s \leq^A t$, if

- $L(s) = L(t)$, and
- for all $a \in \mathtt{Act}_{\mathcal{I}}$ there exists $a' \in \mathtt{Act}_{\mathcal{I}}$ such that $\delta(s, a) \leq^A_{Sm} \delta(t, a')$, where $\delta(s, a)$ is the "curried" transition function defined by $\{s' \in S \mid \exists b \in \mathtt{Act}_{\mathcal{II}} : \delta(s, \langle a, b\rangle) = s'\}$.

Regarding the above definition of $\leq^A$, we have the following observations. First, we do not require a and a' to be the same action, as only the enforced outcomes are considered for establishing an A-simulation. Second, only *deterministic* strategies are used in the original A-simulation in [7]. Intuitively, on state t action a' enforces a *more restrictive outcome* than action a enforces on state s, as shown by the Smyth-ordered relation $\leq^A_{Sm}$: for every $b' \in \mathtt{Act}_{\mathcal{II}}$ there exists $b \in \mathtt{Act}_{\mathcal{II}}$ such that $\delta(s, \langle a, b\rangle) \leq^A \delta(t, \langle a', b'\rangle)$.

Previously, Zhang and Pang have extended A-simulation to probabilistic alternating simulation (PA-simulation) in PGS [8] and proposed an algorithm for computing the largest PA-simulation [17]. Their definition requires lifting of the simulation relation to derive a relation on distributions of states.

Definition 5. *Given a PGS $\langle S, s_0, L, \mathtt{Act}, \delta\rangle$, a binary relation $\mathcal{R} \subseteq S \times S$ is a* probabilistic alternating simulation *(PA-simulation) if whenever $s \mathrel{\mathcal{R}} t$, we have*

- *$L(s) = L(t)$, and*
- *for all $\pi_1 \in \Pi_{\mathcal{I}}$, there exists $\pi_2 \in \Pi_{\mathcal{I}}$, such that $\widetilde{\delta}(s, \pi_1) \ \overline{\mathcal{R}}_{Sm} \ \widetilde{\delta}(t, \pi_2)$, where $\widetilde{\delta}(s, \pi) = \{\Delta \in \mathcal{D}(S) \mid \exists \pi' \in \Pi_{\mathcal{II}} : \widetilde{\delta}(s, \langle \pi, \pi'\rangle) = \Delta\}$.*

State s is PA-similar to t, written as $s \sqsubseteq t$, if there is a PA-simulation $\mathcal{R}$ with $s \mathrel{\mathcal{R}} t$.

If state s PA-simulates state t and t PA-simulates s, we say s and t are *PA-simulation equivalent*, which is written $s \simeq t$.

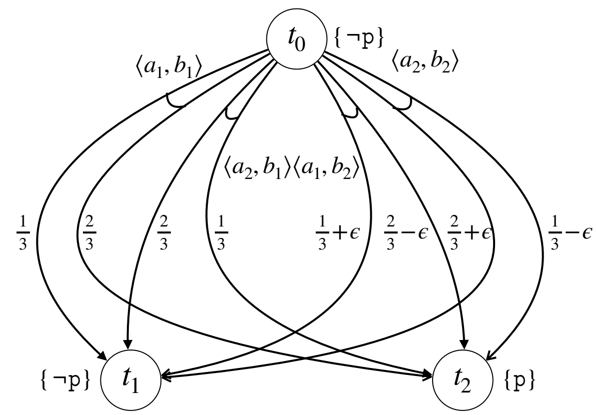

Figure 4: A probabilistic game structure $\mathcal{G}_2$

Example 4. *We define another three-state PGS in Figure 4 where the state space is* $\{t_0, t_1, t_2\}$*,* $\mathtt{Act}_{\mathcal{I}} = \{a_1, a_2\}$*, and* $\mathtt{Act}_{\mathcal{II}} = \{b_1, b_2\}$*. Again,* $\epsilon > 0$ *is a tiny positive real number with the same value as in Example 2. The transition function is now given as follows.*

- $\delta(t_0, \langle a_1, b_1\rangle) = \Delta_2$*, where* $\Delta_2(t_1) = \frac{1}{3}$ *and* $= \Delta_2(t_2) = \frac{2}{3}$.
- $\delta(t_0, \langle a_2, b_1\rangle) = \Delta_3$*, where* $\Delta_3(t_1) = \frac{2}{3}$ *and* $= \Delta_3(t_2) = \frac{1}{3}$.
- $\delta(t_0, \langle a_1, b_2\rangle) = \Delta_2'$*, where* $\Delta_2'(t_1) = \frac{1}{3} + \epsilon$ *and* $\Delta_2'(t_2) = \frac{2}{3} - \epsilon$.
- $\delta(t_0, \langle a_2, b_2\rangle) = \Delta_3'$*, where* $\Delta_3'(t_1) = \frac{2}{3} + \epsilon$ *and* $\Delta_3'(t_2) = \frac{1}{3} - \epsilon$.

There are only self-transitions from all states t_1 *and* t_2*. In this PGS, from* t_0*, if player* $\mathcal{I}$ *chooses action* a_1*, then the destination distribution will be either* Δ_2 *which satisfies* $\Delta_2(t_1) = \frac{1}{3}$ *and* $\Delta_2(t_2) = \frac{2}{3}$ *if player* $\mathcal{II}$ *performs* b_1*, or* Δ_2' *which is very close to* Δ_2 *if player* $\mathcal{II}$ *performs* b_2*. If player* $\mathcal{I}$ *chooses action* a_2*, then the destination distribution will be* Δ_3 *or* Δ_3' *which is very close to* Δ_3*. That is,* $\mathcal{G}_2$ *is designed to make player* $\mathcal{I}$ *the dominant player in this game.*

Suppose we take $\mathcal{G}_1$ and $\mathcal{G}_2$ combined as a single PGS, it is easy to establish $s_1 \sqsubseteq t_1$ and $s_2 \sqsubseteq t_2$ as all these four states have no outgoing transitions, $L(s_1) = L(t_1)$, and $L(s_2) = L(t_2)$. Here we show that $s_0 \sqsubseteq t_0$. Since from

s_0 player $\mathcal{I}$ has only a single mixed action $a_\star$, from t_0 we construct a mixed action σ of player $\mathcal{I}$, satisfying $\sigma(t_0, a_1) = \frac{1}{2}$ and $\sigma(t_0, a_2) = \frac{1}{2}$. Next we need to establish $\widetilde{\delta}(s_0, a_\star) \overline{\sqsubseteq}_{Sm} \widetilde{\delta}(t_0, \sigma)$.

- Suppose the action from player $\mathcal{II}$ in state t_0 is b_1, and let $\widetilde{\delta}(t_0, \langle \sigma, b_1 \rangle) = \Theta_1$, then we have $\Theta_1(t_1) = \frac{1}{2}$ and $\Theta_1(t_2) = \frac{1}{2}$. In this case we let player $\mathcal{II}$'s action from s_0 also be b_1, thus $\widetilde{\delta}(s_0, \langle a_\star, b_1 \rangle) = \Theta_2$, which satisfies $\Theta_2(s_1) = \frac{1}{2}$ and $\Theta_2(s_2) = \frac{1}{2}$. Given $s_1 \sqsubseteq t_1$ and $s_2 \sqsubseteq t_2$, this consequently establishes $\Theta_2 \overline{\sqsubseteq} \Theta_1$.

- Suppose the action from player $\mathcal{II}$ in state t_0 is b_2, and let $\widetilde{\delta}(t_0, \langle \sigma, b_2 \rangle) = \Theta_3$, then we have $\Theta_3(t_1) = \frac{1}{2} + \epsilon$ and $\Theta_3(t_2) = \frac{1}{2} - \epsilon$. In this case we let player $\mathcal{II}$'s action from s_0 be b_2 as well, thus $\widetilde{\delta}(s_0, \langle a_\star, b_2 \rangle) = \Theta_4$, which satisfies $\Theta_4(s_1) = \frac{1}{2} + \epsilon$ and $\Theta_4(s_2) = \frac{1}{2} - \epsilon$. Again, as $s_1 \sqsubseteq t_1$ and $s_2 \sqsubseteq t_2$, this consequently establishes $\Theta_4 \overline{\sqsubseteq} \Theta_3$.

- Suppose in t_0 player $\mathcal{II}$ performs a mixed action that allocates probability p to b_1 and $1 - p$ to b_2 for any $0 < p < 1$, we are able to establish that the same mixed action from s_0 (which allocates probability p to b_1 and $1 - p$ to b_2) produces the same result as in the above cases.

Since $\widetilde{\delta}(s_0, a_\star) \overline{\sqsubseteq}_{Sm} \widetilde{\delta}(t_0, \sigma)$, we have effectively established $s_0 \sqsubseteq t_0$. However, $t_0 \not\sqsubseteq s_0$. This is because if from t_0 player $\mathcal{I}$ performs $a_{1\star}$ which enforces a distribution that gives almost probability $\frac{1}{3}$ to t_1 and almost probability $\frac{2}{3}$ to t_2, there exists no mixed action from s_0 that can enforce a similar outcome. Therefore, intuitively player $\mathcal{I}$ has strictly more behaviour from t_0 than from s_0.

Bisimulation [1] may be the most popular concept for behavioural equivalence in concurrency modelling. Next we define this notion for PGS models.

Definition 6. *Given a PGS $\langle S, s_0, L, \mathtt{Act}, \delta \rangle$, a binary relation $\mathcal{R} \subseteq S \times S$ is a* probabilistic alternating bisimulation *(PA-bisimulation) if whenever $s \; \mathcal{R} \; t$, we have*

- $L(s) = L(t)$, *and*

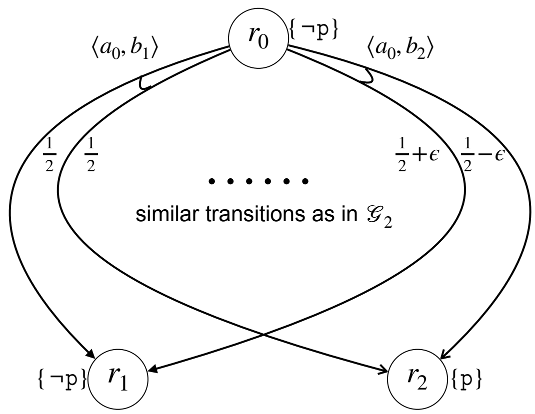

Figure 5: A probabilistic game structure $\mathcal{G}_3$

- *for all* $\pi_1 \in \Pi_{\mathcal{I}}$*, there exists* $\pi_2 \in \Pi_{\mathcal{I}}$*, such that* $\widetilde{\delta}(s,\pi_1)\ \overline{\mathcal{R}}_{Sm}\ \widetilde{\delta}(t,\pi_2)$*, where* $\widetilde{\delta}(s,\pi) = \{\Delta \in \mathcal{D}(S) \mid \exists \pi' \in \Pi_{\mathcal{II}} : \widetilde{\delta}(s,\langle \pi,\pi'\rangle) = \Delta\}$*, and*
- *for all* $\pi_2 \in \Pi_{\mathcal{I}}$*, there exists* $\pi_1 \in \Pi_{\mathcal{I}}$*, such that* $\widetilde{\delta}(s,\pi_1)\ \overline{\mathcal{R}}_{Sm}\ \widetilde{\delta}(t,\pi_2)$*.*

State s *is PA-bisimilar to* t*, written as* $s \approx t$*, if there is a PA-bisimulation* $\mathcal{R}$ *with* $s\ \mathcal{R}\ t$*.*

Example 5. *We introduce a new PGS* $\mathcal{G}_3$ *illustrated in Figure 5 (the omitted transitions are identical to those in* $\mathcal{G}_2$ *depicted in Figure 4), where player* $\mathcal{I}$ *has three actions* a_0*,* a_1 *and* a_2 *available in* r_0*. The transition relations from* r_0 *is*

- $\delta(r_0,\langle a_0,b_1\rangle) = \Theta_1$*, where* $\Theta_1(r_1) = \Theta_1(r_2) = \frac{1}{2}$*.*
- $\delta(r_0,\langle a_0,b_2\rangle) = \Theta_1'$*, where* $\Theta_1'(r_1) = \frac{1}{2}+\epsilon$ *and* $\Theta_1'(r_2) = \frac{1}{2}-\epsilon$*.*
- $\delta(r_0,\langle a_1,b_1\rangle) = \Theta_2$*, where* $\Theta_2(r_1) = \frac{1}{3}$ *and* $= \Theta_2(r_2) = \frac{2}{3}$*.*
- $\delta(r_0,\langle a_1,b_2\rangle) = \Theta_2'$*, where* $\Theta_2'(r_1) = \frac{1}{3}+\epsilon$ *and* $\Theta_2'(r_2) = \frac{2}{3}-\epsilon$*.*
- $\delta(r_0,\langle a_2,b_1\rangle) = \Theta_3$*, where* $\Theta_3(r_1) = \frac{2}{3}$ *and* $\Theta_3(r_2) = \frac{1}{3}$*.*
- $\delta(r_0,\langle a_2,b_2\rangle) = \Theta_3'$*, where* $\Theta_3'(r_1) = \frac{2}{3}+\epsilon$ *and* $\Theta_3'(r_2) = \frac{1}{3}-\epsilon$*.*

One may find that player $\mathcal{I}$ actions a_0, a_1 and a_2 enforce that after a transition, the probabilities to reach state r_1 are within the ranges $[\frac{1}{2}, \frac{1}{2}+\epsilon]$, $[\frac{1}{3}, \frac{1}{3}+\epsilon]$, and $[\frac{2}{3}, \frac{2}{3}+\epsilon]$, respectively. Informally, from r_0, player $\mathcal{I}$ has as much behavioral power as from s_0 and t_0 combined. Similar to showing $s_0 \sqsubseteq t_0$, it is straightforward to establish $r_0 \sqsubseteq t_0$ as well, and this is because the additional behaviour brought by action a_0 from r_0 can be simulated by a mixed action σ from t_0 with $\sigma(t_0, a_1) = \frac{1}{2}$ and $\sigma(t_0, a_2) = \frac{1}{2}$. Moreover, we can further establish $r_0 \approx t_0$, and we leave the details of that relation to the interested reader.

3.1. Approximating Probabilistic Simulation Relations

In the literature, bisimulation (simulation) relation is often defined as an observational equivalence (observational preorder). In an LTS, given $n \in \mathbb{N}$, we say t can "follow" s up to $n+1$ steps, written $s \leq_{n+1} t$, if (1) $s \leq_n t$ and (2) for each step $s \xrightarrow{a} s'$, there exists $t \xrightarrow{a} t'$ and $s' \leq_n t'$. The base relation $\leq_0$ is the universal relation on states.[3] Such a way of definition may provides a few benefits. For example, it allows to define weaker "approximants" for similarity and bisimilarity, which can be applied in scenarios where the "full" simulation relations are too strong. Moreover, these approximants can be used as a proof technique [2] for establishing the completeness result for modal characterisation, as we apply in Section 5.2.

For both PA-simulation and PA-bisimulation, we construct approximant relations $\sqsubseteq_n$ and $\approx_n$ for $n \in \mathbb{N}$, where n denotes the number of steps that are required to check for a state to PA-simulate and PA-bisimulate another state.

Definition 7. *Given a PGS $\langle S, s_0, L, \mathtt{Act}, \delta\rangle$, and states $s, t \in S$,*

- *$s \sqsubseteq_0 t$ if $L(s) = L(t)$,*
- *$s \sqsubseteq_{n+1} t$ if (1) $s \sqsubseteq_n t$, and (2) for all $\pi_1 \in \Pi_{\mathcal{I}}$, there exists $\pi_2 \in \Pi_{\mathcal{I}}$, such that $\widetilde{\delta}(s, \pi_1)\, (\overline{\sqsubseteq}_n)_{Sm}\, \widetilde{\delta}(t, \pi_2)$.*

Definition 8. *Given a PGS $\langle S, s_0, L, \mathtt{Act}, \delta\rangle$, and states $s, t \in S$,*

[3] The simulation relation is $\cap_{n\in\mathbb{N}} \leq_n$ provided that the underlying LTS is image-finite.

- *$s \approx_0 t$ if $L(s) = L(t)$,*

- *$s \approx_{n+1} t$ if (1) $s \approx_n t$, (2) for all $\pi_1 \in \Pi_{\mathcal{I}}$, there exists $\pi_2 \in \Pi_{\mathcal{I}}$, such that $\widetilde{\delta}(s, \pi_1)\,(\overline{\approx}_n)_{Sm}\,\widetilde{\delta}(t, \pi_2)$, and (3) for all $\pi_2 \in \Pi_{\mathcal{I}}$, there exists $\pi_1 \in \Pi_{\mathcal{I}}$, such that $\widetilde{\delta}(s, \pi_1)\,(\overline{\approx}_n)_{Sm}\,\widetilde{\delta}(t, \pi_2)$.*

Given both S and `Act` are finite in a PGS, we are going to show that $\sqsubseteq = \bigcap_{n\in\mathbb{N}} \sqsubseteq_n$ and $\approx = \bigcap_{n\in\mathbb{N}} \approx_n$, as follows.

Lemma 3. *Given a PGS $\langle S, s_0, L, \mathtt{Act}, \delta\rangle$,*

1. *Given a PA-simulation $\sqsubseteq$, there exists $n \in \mathbb{N}$ such that $\sqsubseteq_n = \sqsubseteq$,*
2. *Given a PA-bisimulation $\approx$, there exists $n \in \mathbb{N}$ such that $\approx_n = \approx$.*

Proof: We prove case (1) for PA-simulation, and then showing the case (2) for PA-bisimulation is just similar.

Since for PA-simulation $\sqsubseteq$, we have $\sqsubseteq_{i+1} \subseteq \sqsubseteq_i$ for all $i \in \mathbb{N}$ by Definition 7, and given S being finite, we can always find $n \in \mathbb{N}$ such that $\sqsubseteq_n = \sqsubseteq_{n+1}$. We need to show $\sqsubseteq_n = \sqsubseteq$. Let $s \sqsubseteq_n t$, then we also have $s \sqsubseteq_{n+1} t$ (given $\sqsubseteq_n = \sqsubseteq_{n+1}$). Following Definition 7, we establish the following cases.

- $L(s) = L(t)$,

- for all $\pi_1 \in \Pi_{\mathcal{I}}$, there exists $\pi_2 \in \Pi_{\mathcal{I}}$, such that
  $\widetilde{\delta}(s, \pi_1)\,(\overline{\sqsubseteq}_n)_{Sm}\,\widetilde{\delta}(t, \pi_2)$.

This effectively shows that $\sqsubseteq_n$ is a PA-simulation, i.e., $\sqsubseteq_n \subseteq \sqsubseteq$.

Regarding the claim that $\sqsubseteq \subseteq \sqsubseteq_n$, it can be established by a straightforward induction which proves that $\sqsubseteq \subseteq \sqsubseteq_i$ for all $i \in \mathbb{N}$. Therefore, $\sqsubseteq_n = \sqsubseteq$. $\square$

3.2. Lifted PA-Simulation on Distributions

Now we extend the function $\widetilde{\delta}$ to handle transitions from distributions to distributions. Formally, given a distribution $\Delta \in \mathcal{D}(S)$, $\pi_1 \in \Pi_{\mathcal{I}}$ and $\pi_2 \in \Pi_{\mathcal{II}}$, for all $s \in S$, we define $\widetilde{\delta}(\Delta, \langle\pi_1, \pi_2\rangle)(s) = \sum_{t\in\lceil\Delta\rceil} \Delta(t)\cdot\widetilde{\delta}(t, \langle\pi_1, \pi_2\rangle)(s)$. Informally, from distribution Δ, if player $\mathcal{I}$ performs mixed action π_1 and player $\mathcal{II}$ performs mixed action π_2, then the system will make a transition to $\widetilde{\delta}(\Delta, \langle\pi_1, \pi_2\rangle)$.

Note that during this transition, player $\mathcal{I}$ strategy π_1 and player $\mathcal{II}$ strategy π_2 are applied on all states in the support of distribution Δ. Given $t \in \lceil\Delta\rceil$, the destination distribution $\widetilde{\delta}(t, \langle\pi_1, \pi_2\rangle)$ is then joined (with the other distributions) according to the weight of t in Δ. For better readability, sometimes we write $\Delta \xrightarrow{\pi_1,\pi_2} \Theta$ if $\Theta = \widetilde{\delta}(\Delta, \langle\pi_1, \pi_2\rangle)$.

Since the notion of PA-simulation in Definition 5 is defined as a relation on states, in the following we show that the lifted PA-simulation is also a *simulation* on distributions over the states. This is later used as a stepping stone to our soundness result.

In Definition 5, given $s \sqsubseteq t$ and π_1 on s, we focus on the construction of a mixed action π_2 on t which "simulates" behaviour of π_1 on s. When dealing with a relation on distributions (e.g., showing $\Delta \mathrel{\overline{\sqsubseteq}} \Theta$), we may construct a mixed action for each state in the support of Θ, before combing all the mixed actions (for establishment of the simulation relation) according to the weights of states in the support of Θ. Therefore, we study the properties of "splitting" and "merging" of mixed actions. Similar to the way of treating distributions, we allow a linear combination of mixed actions. Given a list of mixed actions $\langle\pi_i\rangle_{i\in I}$ (of player $\mathcal{I}$), and $\langle p_i\rangle_{i\in I}$ satisfying $\sum_{i\in I} p_i = 1$, $\sum_{i\in I} p_i\pi_i$ is a mixed action defined by $\left(\sum_{i\in I} p_i\pi_i\right)(s)(a) = \sum_{i\in I} p_i \cdot (\pi_i(s)(a))$ for all $s \in S$ and $a \in \mathtt{Act}_{\mathcal{I}}$. (In this definition, p_i is the probability of choosing mixed action π_i.) The following two auxiliary lemmas allow us to "split" a mixed action in its role in the next step of game playing.

Lemma 4. *Let $s \in S$, $\pi \in \Pi_{\mathcal{I}}$ and $\sigma = \sum_{i\in I} p_i\sigma_i \in \Pi_{\mathcal{II}}$, then $\widetilde{\delta}(s, \langle\pi, \sigma\rangle) = \sum_{i\in I} p_i \cdot \widetilde{\delta}(s, \langle\pi, \sigma_i\rangle)$.*

Lemma 5. *Let $s \in S$, $\pi = \sum_{i\in I} p_i\pi_i \in \Pi_{\mathcal{I}}$ and $\sigma \in \Pi_{\mathcal{II}}$, then $\widetilde{\delta}(s, \langle\pi, \sigma\rangle) = \sum_{i\in I} p_i \cdot \widetilde{\delta}(s, \langle\pi_i, \sigma\rangle)$.*

Proof: Let $t \in S$, then

$$\begin{array}{rl} & \widetilde{\delta}(s, \langle \pi, \sigma \rangle)(t) \\ = & \sum_{a_1 \in \mathtt{Act}_1} \sum_{a_2 \in \mathtt{Act}_2} \pi(s)(a_1) \cdot \sigma(s)(a_2) \cdot \delta(s, \langle a_1, a_2 \rangle)(t) \\ = & \sum_{a_1 \in \mathtt{Act}_1} \sum_{a_2 \in \mathtt{Act}_2} \pi(s)(a_1) \cdot \sum_{i \in I} p_i \cdot \sigma_i(s)(a_2) \cdot \delta(s, a_1, a_2)(t) \\ = & \sum_{i \in I} p_i \cdot \big(\sum_{a_1 \in \mathtt{Act}_1} \sum_{a_2 \in \mathtt{Act}_2} \pi(s)(a_1) \cdot \sigma_i(s)(a_2) \cdot \delta(s, a_1, a_2)(t) \big) \\ = & \sum_{i \in I} p_i \cdot \widetilde{\delta}(s, \langle \pi, \sigma_i \rangle)(t) \end{array}$$

□

Here we only prove Lemma 5, as the proof of Lemma 4 is similar to that of Lemma 5. These two lemmas show that we can distribute a probability distribution over actions out of a transition operator to the resulting state distribution. The following lemma further allows to shift a linear combination from the source distribution to the destination distribution of a PGS transition, if the players' strategies do not change.

Lemma 6. *Let $\Delta \in \mathcal{D}(S)$ with $\Delta = \sum_{i \in I} p_i \Delta_i$, $\pi \in \Pi_{\mathcal{I}}$ and $\sigma \in \Pi_{\mathcal{II}}$, then we have $\widetilde{\delta}(\Delta, \langle \pi, \sigma \rangle) = \sum_{i \in I} p_i \cdot \widetilde{\delta}(\Delta_i, \langle \pi, \sigma \rangle)$.*

Proof: Let $t \in S$, then

$$\begin{array}{rl} & \sum_{i \in I} p_i \cdot \widetilde{\delta}(\Delta_i, \langle \pi, \sigma \rangle)(t) \\ = & \sum_{i \in I} p_i \cdot \sum_{s \in S} \Delta_i(s) \cdot \widetilde{\delta}(s, \langle \pi, \sigma \rangle)(t) \\ = & \sum_{s \in S} \sum_{i \in I} p_i \Delta_i(s) \cdot \widetilde{\delta}(s, \langle \pi, \sigma \rangle)(t) \\ = & \sum_{s \in S} \Delta(s) \cdot \widetilde{\delta}(s, \langle \pi, \sigma \rangle)(t) \\ = & \widetilde{\delta}(\Delta, \langle \pi, \sigma \rangle)(t) \end{array}$$

□

The next lemma shows another property of PA-simulation. If two distributions (Δ and Θ) can be split into distributions of smaller weights (i.e., $\sum_{i \in I} p_i \cdot \Delta_i$ and $\sum_{i \in I} p_i \cdot \Theta_i$), then by applying the corresponding strategies ($\langle \pi_1, \sigma_1 \rangle$ and $\langle \pi_2, \sigma_2 \rangle$), if PA-simulation can be established for all resulting distributions (i.e., $\widetilde{\delta}(\Delta_i, \langle \pi_1, \sigma_1 \rangle) \sqsubseteq \widetilde{\delta}(\Theta_i, \langle \pi_2, \sigma_2 \rangle)$), then PA-simulation can be established for the resulting combined distributions as organized by the same index set.

Lemma 7. *Let $\Delta, \Theta \in \mathcal{D}(S)$ with $\Delta = \sum_{i \in I} p_i \Delta_i$ and $\Theta = \sum_{i \in I} p_i \Theta_i$, $\pi_1, \pi_2 \in \Pi_{\mathcal{I}}$ and $\sigma_1, \sigma_2 \in \Pi_{\mathcal{II}}$. If $\widetilde{\delta}(\Delta_i, \langle \pi_1, \sigma_1 \rangle) \mathrel{\overline{\sqsubseteq}} \widetilde{\delta}(\Theta_i, \langle \pi_2, \sigma_2 \rangle)$ for all $i \in I$, then $\widetilde{\delta}(\Delta, \langle \pi_1, \sigma_1 \rangle) \mathrel{\overline{\sqsubseteq}} \widetilde{\delta}(\Theta, \langle \pi_2, \sigma_2 \rangle)$.*

Proof: By assumption, $\widetilde{\delta}(\Delta_i, \langle \pi_1, \sigma_1 \rangle) \mathrel{\overline{\sqsubseteq}} \widetilde{\delta}(\Theta_i, \langle \pi_2, \sigma_2 \rangle)$ for all $i \in I$. By Lemma 1, we have $\sum_{i \in I} p_i \cdot \widetilde{\delta}(\Delta_i, \langle \pi_1, \sigma_1 \rangle) \mathrel{\overline{\sqsubseteq}} \sum_{i \in I} p_i \cdot \widetilde{\delta}(\Theta_i, \langle \pi_2, \sigma_2 \rangle)$.

After that, we apply Lemma 6 on both sides of "$\overline{\sqsubseteq}$" to push the linear combination of the individual distributions (i.e., $\widetilde{\delta}(\Delta_i, \langle \pi_1, \sigma_1 \rangle)$ and $\widetilde{\delta}(\Theta_i, \langle \pi_2, \sigma_2 \rangle)$ for each i) into their corresponding start distributions (i.e., $\widetilde{\delta}(\Delta, \langle \pi_1, \sigma_1 \rangle)$ and $\widetilde{\delta}(\Theta, \langle \pi_2, \sigma_2 \rangle)$), which gives $\widetilde{\delta}(\Delta, \langle \pi_1, \sigma_1 \rangle) \mathrel{\overline{\sqsubseteq}} \widetilde{\delta}(\Theta, \langle \pi_2, \sigma_2 \rangle)$. □

Lemma 7 allows to merge the simulation by component distributions on both sides of the relation. The next auxiliary lemma states that given a PA-simulation on states, the lifted PA-simulation on distributions of states can be treated as a *simulation* via mixed actions of player $\mathcal{I}$ and player $\mathcal{II}$.

Lemma 8. *Let $\mathcal{G} = \langle S, s_0, \mathcal{L}, \mathtt{Act}, \delta \rangle$ be a PGS, and $\sqsubseteq$ be a PA-simulation relation for $\mathcal{G}$. Given $\Delta \mathrel{\overline{\sqsubseteq}} \Theta$, for all player $\mathcal{I}$ mixed actions π_1, there exists a mixed action π_2, such that $\widetilde{\delta}(\Delta, \pi_1) \mathrel{\overline{\sqsubseteq}_{Sm}} \widetilde{\delta}(\Theta, \pi_2)$.*

This can be proved by splitting distributions on both sides, and then merge related components to form distributions on both sides of the lifted relation, applying previous lemmas.

Proof: By definition there exists a weight function w, such that for all states $s, t \in S$, we have $w(s,t) > 0$ implies $s \sqsubseteq t$. For each pair of states $s, t \in S$ with $w(s,t) > 0$, we have a mixed action $\pi_{s,t}$ such that $\delta(s, \pi_1) \mathrel{\overline{\sqsubseteq}_{Sm}} \delta(t, \pi_{s,t})$. We construct a mixed action $\pi_2 = \sum_{s,t \in S : w(s,t) > 0} w(s,t) \pi_{s,t}$, and show that $\widetilde{\delta}(\Delta, \pi_1) \mathrel{\overline{\sqsubseteq}_{Sm}} \widetilde{\delta}(\Theta, \pi_2)$.

Let $\sigma_2 \in \Pi_{\mathcal{II}}$, and we show that there exists a mixed action $\sigma_1 \in \Pi_{\mathcal{II}}$ such that $\widetilde{\delta}(\Delta, \langle \pi_1, \sigma_1 \rangle) \mathrel{\overline{\sqsubseteq}} \widetilde{\delta}(\Theta, \langle \pi_2, \sigma_2 \rangle)$. Since for each pair of states $s, t \in S$ with $w(s,t) > 0$, we have $\widetilde{\delta}(s, \pi_1) \mathrel{\overline{\sqsubseteq}_{Sm}} \widetilde{\delta}(t, \pi_{s,t})$, there exists $\sigma_{s,t} \in \Pi_{\mathcal{II}}$ such that $\widetilde{\delta}(s, \langle \pi_1, \sigma_{s,t} \rangle) \mathrel{\overline{\sqsubseteq}} \widetilde{\delta}(t, \langle \pi_{s,t}, \sigma_2 \rangle)$. Define $\sigma_1 = \sum_{s,t \in S : w(s,t) > 0} w(s,t) \sigma_{s,t}$. By

Lemma 1, we have

$$\sum_{s,t\in S, w(s,t)>0} w(s,t)\cdot \widetilde{\delta}(s,\langle \pi_1, \sigma_{s,t}\rangle) \quad \overline{\sqsubseteq} \quad \sum_{s,t\in S, w(s,t)>0} w(s,t)\cdot \widetilde{\delta}(t,\langle \pi_{s,t}, \sigma_2\rangle)$$

Applying Lemma 4 on the LHS of the above equation and Lemma 5 on the RHS, we have $\widetilde{\delta}(s,\langle \pi_1,\sigma_1\rangle)\,\overline{\sqsubseteq}\,\widetilde{\delta}(t,\langle \pi_2,\sigma_2\rangle)$. Then by Lemma 7, we join the distributions according to the probabilities given by the weight function, and get $\widetilde{\delta}(\Delta,\langle \pi_1,\sigma_1\rangle)\,\overline{\sqsubseteq}\,\widetilde{\delta}(\Theta,\langle \pi_2,\sigma_2\rangle)$. □

Similarly, we have the following results for PA-bisimulation.

Lemma 9. *Let $\mathcal{G} = \langle S, s_0, \mathcal{L}, \texttt{Act}, \delta\rangle$ be a PGS, and $\approx$ be a PA-bisimulation relation for $\mathcal{G}$. Given $\Delta \,\overline{\approx}\, \Theta$,*

- *for all player $\mathcal{I}$ mixed actions π_1, there exists a player $\mathcal{I}$ mixed action π_2, such that $\widetilde{\delta}(\Delta,\pi_1)\,\overline{\approx}_{Sm}\,\widetilde{\delta}(\Theta,\pi_2)$,*
- *for all player $\mathcal{I}$ mixed actions π_2, there exists a player $\mathcal{I}$ mixed action π_1, such that $\widetilde{\delta}(\Delta,\pi_1)\,\overline{\approx}_{Sm}\,\widetilde{\delta}(\Theta,\pi_2)$.*

We also have the following results for the approximants of PA-simulation and PA-bisimulation, following the same proof strategy as for Lemma 8.

Corollary 1. *Let $\mathcal{G} = \langle S, s_0, \mathcal{L}, \texttt{Act}, \delta\rangle$ be a PGS, and $n \in \mathbb{N}$.*

1. *Given $\Delta \,\overline{\sqsubseteq}_{n+1}\, \Theta$, for all player $\mathcal{I}$ mixed actions π_1, there exists a player $\mathcal{I}$ mixed action π_2, such that $\widetilde{\delta}(\Delta,\pi_1)\,(\overline{\sqsubseteq}_n)_{Sm}\,\widetilde{\delta}(\Theta,\pi_2)$.*
2. *Given $\Delta \,\overline{\approx}_{n+1}\, \Theta$, then*
   (1) for all player $\mathcal{I}$ mixed actions π_1, there exists a player $\mathcal{I}$ mixed action π_2, such that $\widetilde{\delta}(\Delta,\pi_1)\,(\overline{\approx}_n)_{Sm}\,\widetilde{\delta}(\Theta,\pi_2)$,
   (2) for all player $\mathcal{I}$ mixed actions π_2, there exists a player $\mathcal{I}$ mixed action π_1, such that $\widetilde{\delta}(\Delta,\pi_1)\,(\overline{\approx}_n)_{Sm}\,\widetilde{\delta}(\Theta,\pi_2)$.

4. Modal Logics for Probabilistic GS

In the literature, different modal logics have been introduced to characterise process semantics at different levels. Hennessy-Milner logic (HML) [2] is a classical example, and it provides a sound and complete characterisation of bisimulation semantics in image-finite LTS. In other words, two states (or processes) satisfy the same set of HML formulas iff they are bisimilar. For a more comprehensive survey we refer to [18]. For probabilistic systems, there are modal logics proposed and proved to characterise strong and weak probabilistic (bi)simulation in the model of probabilistic automata [3, 4, 5].

In this section we propose a modal logic for PGS that characterises a player's ability to enforce temporal properties. We define a new logic $\mathcal{L}^{\oplus}$ that resembles the logic of Deng et al. [19, 4]. The syntax of the logic $\mathcal{L}^{\oplus}$ is presented below.

$$\varphi ::= \mathtt{p} \mid \neg\varphi \mid \bigwedge_{i\in I}\varphi_i \mid \bigvee_{i\in I}\varphi_i \mid \langle\langle\mathcal{I}\rangle\rangle\varphi \mid [\![\mathcal{I}]\!]\varphi \mid \bigoplus_{j\in J} p_j\varphi_j \mid \bigsqcap_{j\in J}\varphi_j$$

In particular, $\mathtt{p}$ is an atomic formula that belongs to the set $\mathtt{Prop}$. Formula $\bigwedge_{i\in I}\varphi_i$ produces a conjunction, and $\bigvee_{i\in I}\varphi_i$ produces a disjunction, both via a (possibly infinite) index set I. We then derive $\top = \bigwedge_{i\in\emptyset}\varphi_i$ is a formula that is true everywhere, and $\bot = \bigvee_{i\in\emptyset}\varphi_i$ is a formula false everywhere. $\langle\langle\mathcal{I}\rangle\rangle\varphi$ specifies player $\mathcal{I}$'s ability to enforce φ in the next step. $[\![\mathcal{I}]\!]\varphi$ is the dual of $\langle\langle\varphi\rangle\rangle$ which specifies player $\mathcal{I}$'s "inability" to enforce $\neg\varphi$. (Informally, $[\![\mathcal{I}]\!]\varphi$ is equivalent to $\neg\langle\langle\mathcal{I}\rangle\rangle\neg\varphi$.) The probabilistic summation operator $\bigoplus_{j\in J} p_j\varphi_j$ explicitly specifies that a distribution satisfying such a formula should be split according to pre-defined probability weights, each part with weight p_j satisfying sub-formula φ_j. For a summation formula with index set J, we may explicitly write down each component coupled by its weight, such as in the way of $[p_1, \varphi_1] \oplus [p_2, \varphi_2] \oplus \ldots \oplus [p_{|J|}, \varphi_{|J|}]$. The operator $\bigsqcap_{j\in J}\varphi_j$ asserts the existence of a linear interpolation among formulas φ_j. Negation of a formula is interpreted as distributions that do not satisfy the formula. We use $\mathcal{L}^{\oplus}$ to denote the set of modal formulas defined by the above syntax.

The semantics of $\mathcal{L}^{\oplus}$ is presented as follows. The interpretation of each formula is defined as a set of distributions of states in a finite PGS $\mathcal{G} = \langle S, s_0, L, \mathtt{Act}, \delta\rangle$.

- $\{\![\mathtt{p}]\!\} = \{\Delta \in \mathcal{D}(S) \mid \forall s \in \lceil\Delta\rceil : \mathtt{p} \in L(s)\}$;

- $\{\![\neg\varphi]\!\} = \{\Delta \in \mathcal{D}(S) \mid \Delta \notin \{\![\varphi]\!\}\}$;

- $\{\![\bigwedge_{i\in I} \varphi_i]\!\} = \bigcap_{i\in I} \{\![\varphi_i]\!\}$

- $\{\![\bigvee_{i\in I} \varphi_i]\!\} = \bigcup_{i\in I} \{\![\varphi_i]\!\}$;

- $\{\![\langle\!\langle \mathcal{I} \rangle\!\rangle \varphi]\!\} = \{\Delta \in \mathcal{D}(S) \mid \exists \pi_1 \in \Pi_{\mathcal{I}} : \forall \pi_2 \in \Pi_{\mathcal{II}} : \Delta \xrightarrow{\pi_1,\pi_2} \Theta \Longrightarrow \Theta \in \{\![\varphi]\!\}\}$;

- $\{\![[\![\mathcal{I}]\!]\varphi]\!\} = \{\Delta \in \mathcal{D}(S) \mid \forall \pi_1 \in \Pi_{\mathcal{I}} : \exists \pi_2 \in \Pi_{\mathcal{II}} : \Delta \xrightarrow{\pi_1,\pi_2} \Theta \wedge \Theta \notin \{\![\varphi]\!\}\}$;

- $\{\![\bigoplus_{j\in J} p_j \varphi_j]\!\} = \{\Delta \in \mathcal{D}(S) \mid \Delta = \sum_{j\in J} p_j \Delta_j \wedge \forall j \in J : \Delta_j \in \{\![\varphi_j]\!\}\}$;

- $\{\![\bigsqcap_{j\in J} \varphi_j]\!\} = \{\Delta \in \mathcal{D}(S) \mid \exists \{p_j\}_{j\in J} : \sum_{j\in J} p_j = 1 \wedge \Delta = \sum_{j\in J} p_j \Delta_j \wedge \forall j \in J : \Delta_j \in \{\![\varphi_j]\!\}\}$;

Note here we say a distribution Δ satisfies a propositional formula if the formula holds in every state in the support of Δ. The rest of the semantics is mostly self-explained. Formally, given a formula $\varphi \in \mathcal{L}^{\oplus}$ and a distribution Δ, we write $\Delta \models \varphi$ iff $\Delta \in \{\![\varphi]\!\}$, and $\mathcal{F}_{\mathcal{L}^{\oplus}}(\Delta)$ for the set of formulas $\{\varphi \in \mathcal{L}^{\oplus} \mid \Delta \models \varphi\}$, which denotes the set of formulas satisfied by distribution Δ. We further define $\mathcal{L}^{\ominus}$ for the sublogic of $\mathcal{L}^{\oplus}$ by removing disjunction and the $[\![\mathcal{I}]\!]$ modality, and the negation operator (i.e., $\neg\varphi$) is only applied at the propositional level. Later in Section 5, we prove that the modal logic $\mathcal{L}^{\ominus}$ characterises PA-simulation.

Remark. The probabilistic modal logic proposed by Parma and Segala [3] and Hermanns et al. [5] uses a fragment operator $[\varphi]_\alpha$, such that a distribution satisfies $[\varphi]_\alpha$ (i.e., $\Delta \models [\varphi]_\alpha$) iff there exists $\Delta_1, \Delta_2 \in \mathcal{D}(S)$ such that $\Delta = \Delta_1 \oplus_\alpha \Delta_2$ and $\Delta_1 \models \varphi$. Informally, it states that a fragment of Δ with weight at least α satisfies φ. Note that the summation operator of $\mathcal{L}^{\oplus}$ can be used to encode the fragment operator $[\varphi]_\alpha$, in the way that $\Delta \models [\varphi]_\alpha$ iff $\Delta \models [\alpha, \varphi] \oplus [1-\alpha, \top]$. Therefore, a straightforward adaptation of the logic by Parma and Segala [3] and Hermanns et al. [5] does not yield a more expressive logic than $\mathcal{L}^{\oplus}$.

In particular, the semantics of nondeterministic summation $\bigsqcap_{j\in J} \varphi_j$ allows arbitrary linear interpolation among formulas φ_j. It defines a set of distributions, each of which, say Δ, can be split into $\Delta_1, \Delta_2 \ldots$ of weights $p_1, p_2 \ldots$, i.e., with $\Delta_k \in \{[\varphi_k]\}$ for all $k \in J$, and the only constraints for the weights are that $0 \le p_k \le 1$ and they sum up to 1. This formula represents a weaker version of probabilistic summation formula (i.e., $\bigoplus_{j\in J} p_j\varphi_j$), and the nondeterministic summation operator can be later used to "simulate" the nondeterminisic behavior from player II in the proof for modal characterisation of PA-simulation relation. Similar to the way of treating probabilistic summations, one may write down $\bigsqcap_{j\in J}\varphi_j$ by $[\varphi_1] \sqcap [\varphi_2] \sqcap \ldots \sqcap [\varphi_{|J|}]$ given finite index J. The following lemma is straightforward.

Lemma 10. *Let $\varphi = \bigoplus_{j\in J} p_j\varphi_j$ and $\varphi' = \bigsqcap_{j\in J}\varphi_j$, and $\Delta \in \mathcal{D}(S)$. We have $\Delta \models \varphi$ implies $\Delta \models \varphi'$.*

Similar to most of the literature, given a PGS $\mathcal{G}$, we define preorders on the set of states in $\mathcal{G}$ with respect to satisfaction of the modal logics $\mathcal{L}^{\ominus}$ and $\mathcal{L}^{\oplus}$. Given $s \in S$, write $\mathcal{L}^{\ominus}(s)$ for $\{\varphi \in \mathcal{L}^{\ominus} \mid s \models \varphi\}$ and $\mathcal{L}^{\oplus}(s)$ for $\{\varphi \in \mathcal{L}^{\oplus} \mid s \models \varphi\}$

Definition 9. *Given states $s, t \in S$,*

- $s \sqsubseteq^{\mathcal{L}^{\ominus}} t$ *if* $\mathcal{L}^{\ominus}(s) \subseteq \mathcal{L}^{\ominus}(t)$*;*
- $s \sqsubseteq^{\mathcal{L}^{\oplus}} t$ *if* $\mathcal{L}^{\oplus}(s) \subseteq \mathcal{L}^{\oplus}(t)$*.*

If $s \sqsubseteq^{\mathcal{L}^{\ominus}} t$ ($s \sqsubseteq^{\mathcal{L}^{\oplus}} t$) and $t \sqsubseteq^{\mathcal{L}^{\ominus}} s$ ($t \sqsubseteq^{\mathcal{L}^{\oplus}} s$), we write $s \approx^{\mathcal{L}^{\ominus}} t$ ($s \approx^{\mathcal{L}^{\oplus}} t$).

In the following sections we state the main results of the paper.

5. Characterising Probabilistic Alternating Simulation

In this section we prove that the modal logic $\mathcal{L}^{\ominus}$ characterises PA-simulation in the following theorem, and we leave its proof to the following two subsections.

Theorem 1. *Given a PGS $\langle S, s_0, L, \mathtt{Act}, \delta\rangle$, for all $s, t \in S$, $s \sqsubseteq t$ iff $s \sqsubseteq^{\mathcal{L}^{\ominus}} t$.*

5.1. A Soundness Proof

Since Lemma 8 extends the PA-simulation to a binary relation on distributions, we may rather prove a more general soundness result, as follows.

Theorem 2. *Given $\varphi \in \mathcal{L}^{\ominus}$, $\Delta, \Theta \in \mathcal{D}(S)$, and $\Delta \,\overline{\sqsubseteq}\, \Theta$ where $\sqsubseteq$ is a PA-simulation, then $\Delta \models \varphi$ implies $\Theta \models \varphi$.*

Proof: By structural induction on the formula φ.

Base case: As $\Delta \,\overline{\sqsubseteq}\, \Theta$, by definition of lifting, there exists a weight function w, such that for every $t \in \lceil \Theta \rceil$ there is $s' \in \lceil \Delta \rceil$ such that $w(s', t) > 0$ and $s' \sqsubseteq t$; and for every $s \in \lceil \Delta \rceil$ there is $t' \in \lceil \Theta \rceil$ such that $w(s, t') > 0$ and $s \sqsubseteq t'$. We consider the following two cases.

(1) If $\Delta \models \varphi$ and $\varphi = \mathtt{p}$, then for all $s \in \lceil \Delta \rceil$, $\mathtt{p} \in L(s)$. Then $\mathtt{p} \in L(t)$ for all $t \in \lceil \Theta \rceil$ by Definition 5 and the weight function w. Therefore, $\Theta \models \mathtt{p}$.

(2) If $\Delta \models \neg\mathtt{p}$, then there exists $s \in \lceil \Delta \rceil$ such that $s \not\models \mathtt{p}$. By the definition of the weight function, there exists $t \in \lceil \Theta \rceil$ such that $w(s, t) > 0$, and $s \sqsubseteq t$. Therefore $L(s) = L(t)$, i.e., $t \not\models \mathtt{p}$. This proves $\Theta \models \neg\mathtt{p}$.

Induction step: We have the following cases.

- If $\varphi = \bigwedge_{i \in I} \varphi_i$, then for every φ_i, we have $\Delta \models \varphi_i$. By I.H., we have $\Theta \models \varphi_i$ for all i. Therefore, $\Theta \models \bigwedge_{i \in I} \varphi_i$.

- If $\varphi = \bigoplus_{i \in I} p_i \varphi_i$, then by definition, there exist $\langle p_i, \Delta_i \rangle_{i \in I}$ such that $\Delta = \sum_{i \in I} p_i \Delta_i$ and $\Delta_i \models \varphi_i$ for all $i \in I$. By Lemma 2(1), there exist $\langle \Theta_i \rangle_{i \in I}$ such that $\Theta = \sum_{i \in I} p_i \Theta_i$ and $\Delta_i \,\overline{\sqsubseteq}\, \Theta_i$ for all $i \in I$. Then by I.H., $\Theta_i \models \varphi_i$ for all i. Therefore, $\Theta \models \bigoplus_{i \in I} p_i \varphi_i$.

- If $\varphi = \bigsqcap_{i \in I} \varphi_i$, then by definition, there exists $\langle p_i, \Delta_i \rangle_{i \in I}$ such that $\sum_{i \in I} p_i = 1$, $\Delta = \sum_{i \in I} p_i \Delta_i$ and $\Delta_i \models \varphi_i$ for all $i \in I$. Then similar to the above case, by Lemma 2(1), there exist $\langle \Theta_i \rangle_{i \in I}$ such that $\Theta = \sum_{i \in I} p_i \Theta_i$ and $\Delta_i \,\overline{\sqsubseteq}\, \Theta_i$ for all $i \in I$. Then by I.H., $\Theta_i \models \varphi_i$ for all i. Therefore, $\Theta \models \bigsqcap_{i \in I} \varphi_i$ by definition.

- If $\varphi = \langle\!\langle \mathcal{I} \rangle\!\rangle \psi$, then there exists a player $\mathcal{I}$ mixed actions π_1 such that for all player $\mathcal{II}$ mixed actions σ_1, $\Delta \xrightarrow{\pi_1, \sigma_1} \Delta'$ and $\Delta' \models \psi$. By Lemma 8, there exists a player $\mathcal{I}$ mixed action π_2 such that $\widetilde{\delta}(\Delta, \pi_1) \,\overline{\sqsubseteq}_{Sm} \widetilde{\delta}(\Theta, \pi_2)$. Therefore, for all player $\mathcal{II}$ mixed actions σ_2 there exists a player $\mathcal{II}$ strategy σ_1', such that $\Delta \xrightarrow{\pi_1, \sigma_1'} \Delta''$, $\Theta \xrightarrow{\pi_2, \sigma_2} \Theta'$, and $\Delta'' \,\overline{\sqsubseteq}\, \Theta'$. Since

$\Delta'' \models \psi$, by I.H., $\Theta' \models \psi$. Now we have that π_2 is the player $\mathcal{I}$ mixed action showing $\Theta \models \langle\!\langle \mathcal{I} \rangle\!\rangle \psi$.

$\square$

5.2. A Completeness Proof

The completeness is proved by making use of the approximant relations of $\sqsubseteq$ and $\sqsubseteq^{\mathcal{L}^{\ominus}}$. Similar to the relations $\sqsubseteq_n$ with $n \in \mathbb{N}$ as in Definition 7, where n denotes the number of steps that are required to check for a state to simulate another, we define $\sqsubseteq_n^{\mathcal{L}^{\ominus}}$ by placing restriction on formulas in $\mathcal{L}^{\ominus}$ with *size* up to n. Then we prove that the relation $\sqsubseteq_n^{\mathcal{L}^{\ominus}}$ is contained in $\sqsubseteq_n$ for all $n \in \mathbb{N}$.

Definition 10. *Let $\mathcal{L}_0^{\ominus}$ be the set of formulas constructed by using only $\mathtt{p}$ (where $\mathtt{p} \in \mathit{Prop}$) and $\bigwedge_{i \in I} \varphi_i$. For $n \in \mathbb{N}$, a formula $\varphi \in \mathcal{L}_{n+1}^{\ominus}$ if either $\varphi \in \mathcal{L}_n^{\ominus}$ or φ is a conjunction of formulas of the form $\langle\!\langle \mathcal{I} \rangle\!\rangle \bigsqcap_{i \in I} \bigoplus_{j \in J} p_j \varphi_{i,j}$, where each $\varphi_{i,j} \in \mathcal{L}_n^{\ominus}$.*

Intuitively, formulas in $\mathcal{L}_n^{\ominus}$ require n steps of transitions (for player $\mathcal{I}$) to enforce. Given states $s, t \in S$, we write $s \sqsubseteq_n^{\mathcal{L}^{\ominus}} t$, if for all $\varphi \in \mathcal{L}_n^{\ominus}$, $s_\star \models \varphi$ implies $t_\star \models \varphi$.

Before starting the completeness proof, we define formulas that characterise properties of the game states. Let $s \in S$, the 0-characteristic formula for s is $\phi_s^0 = \bigwedge \{\mathtt{p} \mid \mathtt{p} \in L(s)\}$. Plainly, the level 0-characterisation considers only propositional formulas. For a distribution, we specify the characteristic formulas for the states in its support proportional to weights. The 0-characteristic formula ϕ_Δ^0 for distribution Δ is $\bigoplus_{t \in \lceil \Delta \rceil} \Delta(t) \cdot \phi_t^0$. Given all n-characteristic formulas defined, the $(n+1)$-characteristic formula ϕ_s^{n+1} for state s is $\bigwedge_{\pi \in \mathcal{D}(\mathtt{Act}_{\mathcal{I}})} \langle\!\langle \mathcal{I} \rangle\!\rangle \bigsqcap_{b \in \mathtt{Act}_{\mathcal{II}}} \phi_{\Delta_{\pi,b}}^n$, where $s_\star \xrightarrow{\pi, b_\star} \Delta_{\pi,b}$. Similarly, an n-characteristic formula ϕ_Δ^{n+1} for distribution Δ is $\bigoplus_{t \in \lceil \Delta \rceil} \Delta(t) \cdot \phi_t^{n+1}$.

Obviously every state or distribution satisfies its own characteristic formula, and the following lemma can be proved by induction on $n \in \mathbb{N}$.

Lemma 11. *For all $\Delta \in \mathcal{D}(S)$, $\Delta \models \phi_\Delta^n$ for all $n \in \mathbb{N}$.*

Lemma 12. *For all states $s, t \in S$ and $n \in \mathbb{N}$, $s \sqsubseteq_n^{\mathcal{L}^{\ominus}} t$ implies $s \sqsubseteq_n t$.*

Proof: Since for each $n \in \mathbb{N}$, we have $s_\star \models \phi_s^n$ by Lemma 11. Let $s \sqsubseteq_n^{\mathcal{L}^\ominus} t$, then $t_\star \models \phi_s^n$. Therefore, we need to prove $t_\star \models \phi_s^n$ implies $s \sqsubseteq_n t$. We proceed by induction on the level of approximation n to show that $s \sqsubseteq_n t$.

We first show that it is equivalent to have this pattern of reasoning working over distributions (as well as over states). Suppose for all $s, t \in S$, we have $s \sqsubseteq_n^{\mathcal{L}^\ominus} t$ implies $t_\star \models \phi_s^n$. Given two distributions $\Delta, \Theta \in \mathcal{D}(S)$, assume that $\Delta \overline{\sqsubseteq}_n^{\mathcal{L}^\ominus} \Theta$. Then there exists a weight function w, such that $\Delta = \sum_{s \in \lceil\Delta\rceil, t \in \lceil\Theta\rceil : w(s,t)>0} w(s,t) \cdot s_\star$, and $\Theta = \sum_{s \in \lceil\Delta\rceil, t \in \lceil\Theta\rceil : w(s,t)>0} w(s,t) \cdot t_\star$, and $s \sqsubseteq_n^{\mathcal{L}^\ominus} t$ for all $w(s,t) > 0$. Since ϕ_Δ^n can be written as $\bigoplus_{s \in \lceil\Delta\rceil, t \in \lceil\Theta\rceil : w(s,t)>0} w(s,t) \cdot \phi_s^n$, we must have $\Theta \models \phi_\Delta^n$ as well.

Base case: Given $s \sqsubseteq_0^{\mathcal{L}^\ominus} t$, then $t_\star \models \phi_s^0$, i.e., states s and t agree on all propositional formulas, which implies $L(s) = L(t)$. Therefore, $s \sqsubseteq_0 t$.

Induction step: Assume the condition holds up to level n. Let $s \sqsubseteq_{n+1}^{\mathcal{L}^\ominus} t$, and we need to show that $s \sqsubseteq_{n+1} t$. Taking the $(n+1)$-characteristic formula for s, then by $s \sqsubseteq_{n+1}^{\mathcal{L}^\ominus} t$, we have $t_\star \models \phi_s^{n+1}$, where $\phi_s^{n+1} = \bigwedge_{\pi \in \mathcal{D}(\mathtt{Act}_{\mathcal{I}})} \langle\!\langle \mathcal{I} \rangle\!\rangle \bigsqcap_{b \in \mathtt{Act}_{\mathcal{II}}} \phi_{\Delta_{\pi,b}}^n$. Then for each $\pi \in \mathcal{D}(\mathtt{Act}_{\mathcal{I}})$, $t_\star \models \langle\!\langle \mathcal{I} \rangle\!\rangle \bigsqcap_{b \in \mathtt{Act}_{\mathcal{II}}} \phi_{\Delta_{\pi,b}}^n$. By definition there exists a player $\mathcal{I}$ mixed action π', such that for every player $\mathcal{II}$ mixed action σ, we have $t_\star \xrightarrow{\pi', \sigma} \Theta$ and $\Theta \models \bigsqcap_{b \in \mathtt{Act}_{\mathcal{II}}} \phi_{\Delta_{\pi,b}}^n$ (i.e., π' enforces $\bigsqcap_{b \in \mathtt{Act}_{\mathcal{II}}} \phi_{\Delta_{\pi,b}}^n$). We need to show that $\widetilde{\delta}(s, \pi) \ (\overline{\sqsubseteq}_n)_{Sm} \ \widetilde{\delta}(t, \pi')$.

It suffices to check each "deterministic action" action in $\mathtt{Act}_{\mathcal{II}}$ from t can be followed by a player $\mathcal{II}$ mixed action from s to establish a simulation. Let b' be a player $\mathcal{II}$ action, and $t_\star \xrightarrow{\pi', b'_\star} \Theta$. Since $\Theta \models \bigsqcap_{b \in \mathtt{Act}_{\mathcal{II}}} \phi_{\Delta_{\pi,b}}^n$, there exists a list of probability values $\langle p_c \rangle_{c \in \mathtt{Act}_{\mathcal{II}}}$, such that $\Theta \models \sum_{c \in \mathtt{Act}_{\mathcal{II}}} p_c \phi_{\Delta_{\pi,c}}^n$. Then by definition, we have $\Theta = \sum_{c \in \mathtt{Act}_{\mathcal{II}}} p_c \cdot \Theta_c$, $\sum_{c \in \mathtt{Act}_{\mathcal{II}}} p_c = 1$ and $\Theta_c \models \phi_{\Delta_{\pi,c}}^n$ for all $c \in \mathtt{Act}_{\mathcal{II}}$. In state s, we define a player $\mathcal{II}$ mixed action σ satisfying $\sigma(s)(c) = p_c$ for all $c \in \mathtt{Act}_{\mathcal{II}}$. Then by Lemma 4, we have $\widetilde{\delta}(s, \langle \pi, \sigma \rangle) = \sum_{c \in \mathtt{Act}_{\mathcal{II}}} p_c \cdot \delta(s, \langle \pi, c \rangle) = \Delta_{\pi,\sigma}$. By Lemma 1, it suffices to show $\delta(s, \langle \pi, c \rangle) \overline{\sqsubseteq}_n \Theta_c$ for all $c \in \mathtt{Act}_{\mathcal{II}}$. Since $\Theta_c \models \phi_{\Delta_{\pi,c}}^n$, we have $\Delta_{\pi,c} \overline{\sqsubseteq}_n \Theta_c$ by I.H.. Therefore, $\Delta_{\pi,\sigma} \ \overline{\sqsubseteq} \ \Theta$. Since $b' \in \mathtt{Act}_{\mathcal{II}}$ is arbitrarily chosen, we have $\widetilde{\delta}(s, \pi) \ (\overline{\sqsubseteq}_n)_{Sm} \ \widetilde{\delta}(t, \pi')$, as required. This proves $s \sqsubseteq_{n+1} t$. $\square$

Intuitively, by fixing a mixed strategy from player $\mathcal{I}$, a transition in the

PGS is bounded by deterministic actions from player $\mathcal{II}$, as mimicked in the structure of the characteristic formulas. The way of showing satisfaction of a characteristic formula thus mimics the PA-simulation in the proof of Lemma 12.

Theorem 3. *For all $s, t \in S$, $s \sqsubseteq^{\mathcal{L}^{\ominus}} t$ implies $s \sqsubseteq t$.*

Proof: In a finite state PGS (i.e., the space $S \times S$ is finite) there exists $n \in \mathbb{N}$ such that $\sqsubseteq = \sqsubseteq_n$ by Lemma 3(1). Since $\sqsubseteq^{\mathcal{L}^{\ominus}} \subseteq \sqsubseteq_n^{\mathcal{L}^{\ominus}}$, and $\sqsubseteq_n^{\mathcal{L}^{\ominus}} \subseteq \sqsubseteq_n$ which is by Lemma 12, we have $\sqsubseteq^{\mathcal{L}^{\ominus}} \subseteq \sqsubseteq_n = \sqsubseteq$. □

6. Characterising Probabilistic Alternating Bisimulation

In this section, we prove that PA-bisimulation can be characterised by the modal logic $\mathcal{L}^{\oplus}$. The soundness proof is still by structural induction. Regarding the completeness proof, we prove that for all $n \in \mathbb{N}$, if two states are not $\approx_n$ related, then they are not $\approx_n^{\mathcal{L}^{\oplus}}$ related. Similar to what we have defined for $\mathcal{L}^{\ominus}$ in the last section, we define $\mathcal{L}_0^{\oplus}$ for the set of formulas in $\mathcal{L}^{\oplus}$ which does not contain the strategy modality $\langle\!\langle \mathcal{I} \rangle\!\rangle$ (nor $[\![\mathcal{I}]\!]$[4]). For all $n \in \mathbb{N}$, $\mathcal{L}_{n+1}^{\oplus}$ is the union of $\mathcal{L}_n^{\oplus}$, all formulas in $\{\langle\!\langle \mathcal{I} \rangle\!\rangle \varphi \mid \varphi \in \mathcal{L}_n^{\oplus}\}$, and the closure on the former two sets by using all operators defined for $\mathcal{L}^{\oplus}$ except for the strategy modality $\langle\!\langle \mathcal{I} \rangle\!\rangle$.

In the following, we first prove a stronger soundness result for approximants of PA-bisimulation on distributions rather than on states.

Lemma 13. $\Delta \overline{\approx_n} \Theta$ *implies* $\Delta \overline{\approx_n^{\mathcal{L}^{\oplus}}} \Theta$ *for all* $n \in \mathbb{N}$.

Proof: We prove by induction on n.

Base case: Given $\Delta \overline{\approx_0} \Theta$, there exists a weight function w satisfying (1) $\sum_{s \in \lceil \Delta \rceil} w(s,t) = \Theta(t)$ for all $t \in \lceil \Theta \rceil$, (2) $\sum_{t \in \lceil \Theta \rceil} w(s,t) = \Delta(s)$ for all $s \in \lceil \Delta \rceil$, and (3) $w(s,t)$ implies $L(s) = L(t)$. This effectively establishes $\Delta \overline{\approx_0^{\mathcal{L}^{\oplus}}} \Theta$ with the same weight function w.

[4]Strictly speaking, both disjunction and the $[\![\mathcal{I}]\!]$ modality are syntactic sugars and are expressible by the use of negation.

Induction step: Suppose the above claim is satisfied for k, we prove the case for $k+1$. Let $\Delta\overline{\approx_{k+1}}\Theta$, we show that $\Delta\overline{\approx^{\mathcal{L}^{\oplus}}_{k+1}}\Theta$. We have the following cases.

- Let $\langle\langle\mathcal{I}\rangle\rangle\psi$ where $\psi \in \mathcal{L}^{\oplus}_k$ and $\langle\langle\mathcal{I}\rangle\rangle\psi \in \mathcal{L}^{\oplus}_{k+1}$. Suppose $\Delta \models \langle\langle\mathcal{I}\rangle\rangle\psi$, we show that $\Theta \models \langle\langle\mathcal{I}\rangle\rangle\psi$. Let $\pi_1 \in \Pi_{\mathcal{I}}$ be the strategy that enforces $\langle\langle\mathcal{I}\rangle\rangle\psi$ on Δ. Then by Corollary 1(2), there exists $\pi_2 \in \Pi_{\mathcal{I}}$ such that for all $\sigma_2 \in \Pi_{\mathcal{II}}$, there exists σ_1, such that $\Delta \xrightarrow{\pi_1,\sigma_1} \Delta'$, $\Theta \xrightarrow{\pi_2,\sigma_2} \Theta'$, and $\Delta' \overline{\approx_k} \Theta'$. Since π enforces ψ, we have $\Delta' \models \psi$. By I.H., we have $\Theta' \models \psi$, i.e., π_2 is the strategy that enforces $\langle\langle\mathcal{I}\rangle\rangle\psi$ in Θ. Therefore $\Theta \models \langle\langle\mathcal{I}\rangle\rangle\psi$. Suppose $\Theta \models \langle\langle\mathcal{I}\rangle\rangle\psi$, we can show that $\Delta \models \langle\langle\mathcal{I}\rangle\rangle\psi$ which is a symmetric case.
- The cases where the strategy modality is not used can be proved by structural induction on construction of the formulas in $\mathcal{L}^{\oplus}_{n+1}$, similar to the proof for Theorem 2.

$\square$

To prove the completeness result, we show that if two states s and t are not related by $\approx$, then $s \not\approx^{\mathcal{L}^{\oplus}} t$. We prove this claim for all approximant relations $\approx_n$ and $\approx^{\mathcal{L}^{\oplus}}_n$, in the way that if $s \not\approx_n t$, we are able to construct a formula $\varphi \in \mathcal{L}^{\oplus}_n$ which is only satisfied by s, but not t. We need the following auxiliary lemmas. The first lemma is a direct implication of the definition of $\approx^{\mathcal{L}^{\oplus}}_n$.

Lemma 14. *For all $n \in \mathbb{N}$, $s, t \in S$, $s \not\approx^{\mathcal{L}^{\oplus}}_n t$ then there exists $\varphi \in \mathcal{L}^{\oplus}_n$ such that $s_\star \models \varphi$ and $t_\star \not\models \varphi$.*

We enumerate the equivalence classes of $\approx^{\mathcal{L}^{\oplus}}_n$ as $E_1, E_2 \dots E_m$ such that $E_i \cap E_j = \emptyset$ for all $i < j$ and $\bigcup_{i=1\dots m} E_i = S$. Given a state $s \in E_i$, we can find a formula φ_i such that $s \models \varphi_i$ and $t \not\models \varphi_i$ for all states $t \notin E_i$. In the following we extend the result of Lemma 14 to lifted approximants for PA-bisimulation as a relation on the set of distributions.

Lemma 15. *For all $n \in \mathbb{N}$, $\Delta, \Theta \in \mathcal{D}(S)$, $\Delta \overline{\not\approx^{\mathcal{L}^{\oplus}}_n}\Theta$ then there exists $\varphi \in \mathcal{L}^{\oplus}_n$ such that $\Delta \models \varphi$ and $\Theta \not\models \varphi$.*

Proof: Suppose φ_i is a formula in $\mathcal{L}^{\oplus}_n$ that is satisfied by states in E_i but not satisfied by all states in not E_i, by Lemma 14. Let $\Delta(E_i) = p_i$ for all

$i = 1 \ldots m$. We construct a formula $\varphi_\Delta = \sum_{i=1\ldots m} p_i \varphi_i$. Then we have $\Delta \models \varphi_\Delta$ and $\Theta \not\models \varphi_\Delta$. $\square$

Lemma 16. *For all $n \in \mathbb{N}$, $S_1, S_2 \subseteq \mathcal{D}(S)$, satisfying $\Delta \overline{\not\approx_n^{\mathcal{L}^\oplus}} \Theta$ for all $\Delta \in S_1$ and $\Theta \in S_2$, then there exists $\varphi \in \mathcal{L}_n^\oplus$ such that $\Delta \models \varphi$ for all $\Delta \in S_1$ and $\Theta \not\models \varphi$ for all $\Theta \in S_2$.*

Proof: Let $\Delta \in S_1$, then by Lemma 15, there exists a formula φ_Δ such that $\Delta \models \varphi_\Delta$ and $\Theta \not\models \varphi_\Delta$ for all $\Theta \in S_2$. We construct the formula $\phi = \bigvee_{\Delta \in S_1} \varphi_\Delta$. Then we have $\Delta \models \phi$ for all $\Delta \in S_1$ and $\Theta \not\models \phi$ for all $\Theta \in S_2$. $\square$

Now we apply Lemma 16 in the following result.

Lemma 17. *$s \not\approx_n t$ implies $s \not\approx_n^{\mathcal{L}^\oplus} t$ for all $n \in \mathbb{N}$.*

Proof: We prove by induction on $n \in \mathbb{N}$.
BASE CASE: $s \not\approx_0 t$ implies $L(s) \neq L(t)$. Then w.l.o.g., there exists a proposition $\mathtt{p} \in L(t)$ such that $s_\star \not\models \mathtt{p}$ and $t_\star \models \mathtt{p}$. Therefore, $s \not\approx_0^{\mathcal{L}^\oplus} t$.
INDUCTION STEP: Suppose $s \not\approx_{n+1} t$, there exists $\pi \in \Pi_{\mathcal{I}}$ such that for all $\pi' \in \Pi_{\mathcal{I}}$, we have $\widetilde{\delta}(s, \pi)\, (\overline{\not\approx_n})_{Sm}\, \widetilde{\delta}(t, \pi')$. By definition of Smyth order, for all $\pi' \in \Pi_{\mathcal{I}}$, there exists $\sigma' \in \Pi_{\mathcal{II}}$, such that for all $\sigma \in \Pi_{\mathcal{II}}$, we have (1) $s_\star \xrightarrow{\pi,\sigma} \Delta_{\pi,\sigma}$, (2) $t_\star \xrightarrow{\pi',\sigma'} \Theta_{\pi',\sigma'}$, (3) $\Delta_{\pi,\sigma}\ \ \overline{\not\approx_n}\ \Theta_{\pi',\sigma'}$. Then by I.H., $\Delta_{\pi,\sigma}\ \overline{\not\approx_n{}^{\mathcal{L}^\oplus}}\Theta_{\pi',\sigma'}$.

In the above case, we define a function $f : \Pi_{\mathcal{I}} \to \Pi_{\mathcal{II}}$ which maps the choice of player $\mathcal{I}$ mixed actions (e.g., π') to the corresponding player $\mathcal{II}$ mixed actions (e.g., σ') from state t. Since σ can be any player $\mathcal{II}$ strategy, for all $\Delta \in \widetilde{\delta}(s, \pi)$ and for all $\Theta \in \{\Theta_{\pi',\sigma'} \mid \sigma' = f(\pi')\}$, we have $\Delta\, \overline{\not\approx_n{}^{\mathcal{L}^\oplus}}\Theta$. We construct two sets of distributions $S_1 = \widetilde{\delta}(s, \pi)$ and $S_2 = \{\Theta_{\pi',\sigma'} \mid \sigma' = f(\pi')\}$. Now by Lemma 16, there exists $\varphi \in \mathcal{L}_n^\oplus$ such that $\Delta \models \varphi$ for all $\Delta \in S_1$ and $\Theta \not\models \varphi$ for all $\Theta \in S_2$. Therefore, we construct a formula $\langle\langle \mathcal{I} \rangle\rangle \varphi \in \mathcal{L}_{n+1}^\oplus$, with $s \models \langle\langle \mathcal{I} \rangle\rangle \varphi$ by strategy π and $t \not\models \langle\langle \mathcal{I} \rangle\rangle \varphi$. Therefore $s \not\approx_{n+1}^{\mathcal{L}^\oplus} t$. $\square$

By combining Lemma 13 and Lemma 17, we have established $s \approx_n t$ iff $s \approx_n^{\mathcal{L}^\oplus} t$ for all $n \in \mathbb{N}$. Therefore, by applying Lemma 3(2), we have that $\approx = \bigcap_{n \in \mathbb{N}} \approx_n = \bigcap_{n \in \mathbb{N}} \approx_n^{\mathcal{L}^\oplus} = \approx^{\mathcal{L}^\oplus}$, which proves the following.

Theorem 4. *For all $s, t \in S$, $s \approx t$ iff $s \approx^{\mathcal{L}^\oplus} t$.*

7. Probabilistic Alternating-time Mu-Calculus

Modal logics of finite modality depth are not enough to express temporal requirements such as "something bad never happens". In this section, we extend the logic $\mathcal{L}^{\oplus}$ into a Probabilistic Alternating-time μ-calculus (PAMu), by adding variables and fixpoint operators.

$$\begin{aligned} \varphi ::= \quad & p \mid \neg\varphi \mid \bigwedge_{i\in I}\varphi_i \mid \bigvee_{i\in I}\varphi_i \mid \langle\langle\mathcal{I}\rangle\rangle\varphi \mid [\![\mathcal{I}]\!]\varphi \mid \bigsqcap_{j\in J}\varphi_j \\ & \mid \bigoplus_{j\in J} p_j\varphi_j \mid Z \mid \mu Z.\varphi \mid \nu Z.\varphi \end{aligned}$$

Let the environment $\rho : \mathcal{V} \to \mathcal{P}(\mathcal{D}(S))$ be a mapping from variables in $\mathcal{V}$ to sets of distributions on states, and the semantics of the fixpoint operators of PAMu are defined in the standard way.

- $\{[p]\}_\rho = \{\Delta \in \mathcal{D}(S) \mid \forall s \in \lceil\Delta\rceil : p \in L(s)\}$;
- $\{[\neg\varphi]\}_\rho = \{\Delta \in \mathcal{D}(S) \mid \Delta \notin \{[\varphi]\}\}$;
- $\{[\bigwedge_{i\in I}\varphi_i]\}_\rho = \bigcap_{i\in I}\{[\varphi_i]\}_\rho$;
- $\{[\bigvee_{i\in I}\varphi_i]\} = \bigcup_{i\in I}\{[\varphi_i]\}_\rho$;
- $\{[\langle\langle\mathcal{I}\rangle\rangle\varphi]\}_\rho = \{\Delta \in \mathcal{D}(S) \mid \exists\pi_1 \in \Pi_{\mathcal{I}} : \forall\pi_2 \in \Pi_{\overline{\mathcal{I}}} : \Delta \xrightarrow{\pi_1,\pi_2} \Theta \Longrightarrow \Theta \in \{[\varphi]\}_\rho\}$;
- $\{[[\![\mathcal{I}]\!]\varphi]\}_\rho = \{\Delta \in \mathcal{D}(S) \mid \forall\pi_1 \in \Pi_{\mathcal{I}} : \exists\pi_2 \in \Pi_{\overline{\mathcal{I}}} : \Delta \xrightarrow{\pi_1,\pi_2} \Theta \wedge \Theta \notin \{[\varphi]\}_\rho\}$;
- $\{[\bigoplus_{j\in J} p_j\varphi_j]\}_\rho = \{\Delta \in \mathcal{D}(S) \mid \Delta = \sum_{j\in J} p_j\Delta_j \wedge \forall j \in J : \Delta_j \in \{[\varphi_j]\}_\rho\}$;
- $\{[\bigsqcap_{j\in J}\varphi_j]\}_\rho = \{\Delta \in \mathcal{D}(S) \mid \exists\{p_j\}_{j\in J} : \sum_{j\in J} p_j = 1 \wedge \Delta = \sum_{j\in J} p_j\Delta_j \wedge \forall j \in J : \Delta_j \in \{[\varphi_j]\}_\rho\}$;
- $\{[Z]\}_\rho = \rho(Z)$;
- $\{[\mu Z.\varphi]\}_\rho = \bigcap\{D \subseteq \mathcal{D}(S) \mid \{[\varphi]\}_\rho[Z \mapsto D] \subseteq D\}$;
- $\{[\nu Z.\varphi]\}_\rho = \bigcup\{D \subseteq \mathcal{D}(S) \mid D \subseteq \{[\varphi]\}_\rho[Z \mapsto D]\}$.

The set of closed PAMu formulas are the formulas with all variables bounded and satisfy the syntactic condition that in $\mu X.\varphi$ and $\nu X.\varphi$, the variable X may occur in φ only within the scope of an even number of negations. We define this set of formulas as $\mathcal{L}^{\mu}$, and safely drop the environment ρ for those formulas.

Example 6. *For the rock-paper-scissors game in Figure 1, the property describing that player $\mathcal{I}$ has a strategy to eventually win the game once can be expressed as $\mu Z.\mathsf{win}_{\mathcal{I}} \vee \langle\langle \mathcal{I} \rangle\rangle Z$. This property does not hold. However, player $\mathcal{I}$ has a strategy to eventually win the game with probability almost $\frac{1}{2}$, i.e., the system satisfies $\mu Z.([\frac{1}{2}-\epsilon, \mathsf{win}_{\mathcal{I}}] \oplus [\frac{1}{2}+\epsilon, \top]) \vee \langle\langle \mathcal{I} \rangle\rangle Z$ for arbitrarily small $\epsilon > 0$. We explain the reason why players can only enforce ϵ-optimal strategies in a later part of the section.*

The logic characterisation of the simulation relations discussed in this paper can be extended to PAMu. In particular, we show that PA-bisimulation can be characterised by PAMu, and PA-simulation can be characterized by the fragment of PAMu with disjunction and the $[\![\mathcal{I}]\!]$ modality removed, and negation only applied at the propositional level.

Theorem 5. *Given $\Delta, \Theta \in \mathcal{D}(S)$, then*

1. $\Delta \approx \Theta$ *iff for all $\varphi \in \mathcal{L}^{\mu}$, $\Delta \models \varphi \Longleftrightarrow \Theta \models \varphi$.*
2. $\Delta \,\overline{\sqsubseteq}\, \Theta$ *iff for all $\varphi \in \mathcal{L}^{\mu}$ such that φ does not contain disjunction or the $[\![\mathcal{I}]\!]$ modality, and negation is only allowed at the propositional level of φ, we have that $\Delta \models \varphi \Longrightarrow \Theta \models \varphi$.*

We sketch a proof for PA-bismulation here, and the result for PA-simulation can be shown in a similar way. Since $\mathcal{L}^{\oplus}$ is syntactically a sublogic of $\mathcal{L}^{\mu}$, we only need to show the soundness of PA-bisimulation to the logic $\mathcal{L}^{\mu}$. We apply the classical approach of approximants for Modal Mu-Calculus [20]. Given formulas $\mu Z.\varphi$ and $\nu Z.\varphi$, we define the following.

$$
\begin{array}{ll}
\mu^0 Z.\varphi = \bot & \nu^0 Z.\varphi = \top \\
\mu^{i+1} Z.\varphi = \varphi[Z \mapsto \mu^i Z.\varphi] & \nu^{i+1} Z.\varphi = \varphi[Z \mapsto \nu^i Z.\varphi] \\
\mu^{\omega} Z.\varphi = \bigvee_{i\in\mathbb{N}} \mu^i Z.\varphi & \nu^{\omega} Z.\varphi = \bigwedge_{i\in\mathbb{N}} \nu^i Z.\varphi.
\end{array}
$$

Next we show that the approximants are semantically equivalent to the fixpoint formulas.

Lemma 18. 1. $\{[\mu^\omega Z.\varphi]\} = \{[\mu Z.\varphi]\}$;

2. $\{[\nu^\omega Z.\varphi]\} = \{[\nu Z.\varphi]\}$.

We briefly sketch a proof of Lemma 18(1), and the proof for the other part of the lemma is similar. To show $\{[\mu^\omega Z.\varphi]\} \subseteq \{[\mu Z.\varphi]\}$, we initially have $\{[\mu^0 Z.\varphi]\} = \emptyset \subseteq \{[\mu Z.\varphi]\}$, then by the monotonicity of φ, given $\{[\mu^i Z.\varphi]\} \subseteq \{[\mu Z.\varphi]\}$, we prove $\{[\mu^{i+1} Z.\varphi]\} \subseteq \{[\mu Z.\varphi]\}$ by applying φ on both sides of $\subseteq$. Therefore, $\{[\mu^i Z.\varphi]\} \subseteq \{[\mu Z.\varphi]\}$ for all $i \in \mathbb{N}$, thus $\{[\bigvee_{i\in\mathbb{N}} \mu^i Z.\varphi]\} \subseteq \{[\mu Z.\varphi]\}$. To show $\{[\mu Z.\varphi]\} \subseteq \{[\mu^\omega Z.\varphi]\}$, it is easy to see that $\mu^\omega Z.\varphi$ is a prefixpoint, therefore it contains $\mu Z.\varphi$, the intersection of all prefixpoints.

From Lemma 18, a fixpoint formula can be unfolded to a semantically equivalent (approximant) formula in $\mathcal{L}^\oplus$ with countable conjunction or countable disjunction. Then by the soundness of PA-bisimulation to $\mathcal{L}^\oplus$ (Theorem 4), we get the soundness of PA-bisimulation to the logic $\mathcal{L}^\mu$, as required.

Expressiveness of PAMu. There exist game-based extensions of probabilistic temporal logics, such as the logic PAMC [21] that extends the Alternating-time Mu-Calculus [6], and PATL [22] that extends PCTL [23]. The semantics of both logics are sets of states, rather than sets of distributions. It has also been shown in [21] that PAMC and PATL are incomparable on probabilistic game structures, based on a result showing that PCTL and PμTL are incomparable on Markov chains [24]. Here we make a short comparison between PAMu and those logics.

Distribution formulas of PAMu cannot be expressed by state-based logics. For example, the formula $\langle\langle\mathcal{I}\rangle\rangle[\frac{1}{2}, p] \oplus [\frac{1}{2}, q]$, expressing that player $\mathcal{I}$ has a strategy to enforce in the next move a distribution which has half of its weight satisfying p and the other half satisfying q, cannot be expressed by PATL or PAMC. As the latter two logics have probability values bundled with strategy modalities, a formula such as $\langle\langle\mathcal{I}\rangle\rangle^{\geq\frac{1}{2}} p \wedge \langle\langle\mathcal{I}\rangle\rangle^{\geq\frac{1}{2}} q$ denotes that player $\mathcal{I}$ has a strategy to enforce p with at least probability $\frac{1}{2}$ in the next step and player $\mathcal{I}$

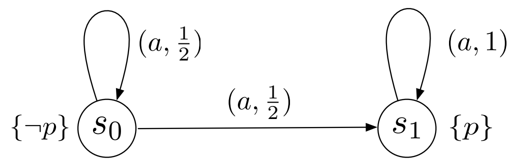

Figure 6: An example for $\langle\!\langle \mathcal{I} \rangle\!\rangle^{\geq 1} \Diamond p$.

also has a *possibly different* strategy to enforce q with at least probability $\frac{1}{2}$ in the next step. However, the resulting states (or distributions) that satisfy p and q may overlap.

The PATL formula $\langle\!\langle \mathcal{I} \rangle\!\rangle^{\geq 1} \Diamond p$ is not expressible by PAMu. Given the PGS in Figure 6 where player $\mathcal{I}$ has action set $\{a\}$ and player $\mathcal{II}$ has action set $\emptyset$. Then it is not difficult to see both s_0 and s_1 satisfies $\langle\!\langle \mathcal{I} \rangle\!\rangle^{\geq 1} \Diamond p$. The closest formula in PAMu is $\mu Z.p \vee \langle\!\langle \mathcal{I} \rangle\!\rangle Z$, but $s_0 \not\models \mu Z.p \vee \langle\!\langle \mathcal{I} \rangle\!\rangle Z$. More precisely, $s_0 \models \mu Z.([\alpha, p] \oplus [1-\alpha, \top]) \vee \langle\!\langle \mathcal{I} \rangle\!\rangle Z$ for all $0 \leq \alpha < 1$. Intuitively, the semantics of the least fixpoint operator in PAMu only track finite number of probabilistic transitions, as starting from s_0, player $\mathcal{I}$ can only reach distributions that satisfy p with probability strictly less than 1 with finite number of steps. Intuitively, $s_{0\star} \xrightarrow{a} [\frac{1}{2}, s_0] \oplus [\frac{1}{2}, s_1] \xrightarrow{a} [\frac{1}{4}, s_0] \oplus [\frac{3}{4}, s_1] \xrightarrow{a} \ldots \xrightarrow{a} [\frac{1}{2^i}, s_0] \oplus [1 - \frac{1}{2^i}, s_1] \ldots$. We shall see that in a finite number of transitions one never reaches $s_{1\star}$ from $s_{0\star}$ with strict probability 1. However, such a restriction may be alleviated in practice, as implemented in PRISM-game [11], ϵ-optimal strategies are synthesized for unbounded reachability properties.

Since the strategy modality in PAMu can only handle the condition for the next step, it cannot express some of the PAMC formulas that goes beyond finite reachability. Based on the semantics of PAMC [21], the above form of PATL formula $\langle\!\langle \mathcal{I} \rangle\!\rangle^{>\alpha} \Diamond p$ can be expressed by the PAMC formula $\langle\!\langle \mathcal{I} \rangle\!\rangle^{>\alpha}(\mu Z.p \vee \langle\!\langle \mathcal{I} \rangle\!\rangle^{\geq 1} Z)$. Therefore, in general, PAMC and PAMu are incomparable.

Example 7. *The authors of [11] proposed a CGS variant of a futures market investor model [25], which studies the interactions between an investor and a stock market. The investor and the market take their decisions simultaneously* *in the CGS model, and the authors show that this does not give any additional benefits to the investor by analysing his or her maximum expected value over a fixed period of time.*[5] *We take this example to demonstrate the expressiveness of PAMu. For instance, the property "it is always possible for the investor to cash in" can be specified with two nested fixpoints as*

$$\nu X.(\mu Y.\mathsf{cashin} \vee \langle\langle \mathit{investor} \rangle\rangle\, Y\,) \wedge \langle\langle \mathit{investor} \rangle\rangle X.$$

Here the greatest fixpoint νX *asserts that the investor is able to enforce the system to keep in a set of states (or a distribution of states taken from this set), such that from any state it is possible to cash in within finite number of transitions which is enforced by the inner least fixpoint* μY*.*

Another interesting property is to check whether the investor has a strategy to ensure a good chance to make a profit. This can be formulated in PAMu with $\frac{1}{2} < \alpha \leq 1$*, as*

$$\mu Z.(\mathsf{cashin} \wedge [\alpha, \mathsf{profit}] \oplus [1-\alpha, \top]) \vee \langle\langle \mathit{investor} \rangle\rangle Z$$

By using the least fixpoint μZ*, this formula asserts that the investor is able to reach a position within finite number of steps where the investor's cash in behaviour is accompanied by a profitable state with at least probability* α*, where* $\alpha > \frac{1}{2}$*.*

8. Related Work

Segala and Lynch [26] introduce a probabilistic simulation relation which preserves probabilistic computation tree logic (PCTL) formulas without negation and existential quantification. Segala introduces the notion of probabilistic forward simulation, which relates states to probability distributions over states and is sound and complete for trace distribution precongruence [27, 28]. Parma and Segala [3] study logic characterisation of probabilistic bisimulation

[5] For details of the model, we refer to [25] and the website `https://www.prismmodelchecker.org`.

for image-finite probabilistic automata. They use a probabilistic extension of the Hennessy-Milner logic which allows countable conjunction and admits a new operator $[\phi]_p$ – a distribution satisfies $[\phi]_p$ if the probability on the set of states satisfying ϕ is at least p, with a sound and complete logic characterisation. Their logic characterisation is both sound and complete. Hermanns et al. [5] further extend this result for image-infinite probabilistic automata.[6] Deng et al. [19, 4] extend the non-probabilistic mu-calculus by adding a few probabilistic operators to derive a probabilistic modal mu-calculus (pMu). A fragment of pMu (without fixed points) has been proved to characterise (strong) probabilistic simulation in finite-state probabilistic automata. Our work extends the above work by enriching it with the concurrent game semantics that are initiated in [6], which is discussed in the following paragraph.

Alur, Henzinger and Kupferman [6] define alternating-time temporal logic (ATL) to generalise CTL for game structures by requiring each path quantifier to be parameterised by a set of agents. GS are more general than LTS, in the sense that they allow both collaborative and adversarial behaviours of individual agents in a system, and ATL can be used to express properties like "a set of agents can enforce a specific outcome of the system". The alternating simulation, which is a natural game-theoretic interpretation of the classical simulation in (deterministic) multi-player games, is introduced in [7]. Logic characterisation of this relation concentrates on a subset of ATL* formulas where negations are only allowed at propositional level and all path quantifiers are parameterised by a predefined set of agents A. This sublogic of ATL* contains all formulas expressing the properties that agents in A can enforce no matter what the other agents do. Alur et al. [7] have proved both soundness and completeness of their characterisation. Comparing with the standard alternating simulation and its logic characterisation, PA-simulation focuses on the extension which allows

[6] Logic characterisation of weak probabilistic bisimulation has been studied in [29], where the logic PCTL* is used. This result is extended to weak probabilistic simulation by Parma [30].

mixed strategies in probabilistic game structures (PGS).

Game structures deal well with systems in which the players execute a *pure strategy*, i.e., a strategy in which the moves are chosen deterministically. However, *mixed strategies*, which are formed by probabilistically combining pure strategies, are necessary for a player to achieve *optimal* rewards [12]. Zhang and Pang [8] extend the notion of game structures to probabilistic game structures (PGS) and introduce notions of simulation that are sound for a fragment of probabilistic alternating-time temporal logic (PATL), a probabilistic extension of ATL.

Fixpoint logics for sets of states in Markov chains and PGS have been studied more recently in [24, 21], and a short comparison is given in Section 7.

Metric-based simulation on game structures have been studied by de Alfaro et al. [31] regarding the probability of winning games whose goals are expressed in quantitative μ-calculus (qMu) [25]. Two states are equivalent if the players can win the same games with the same probability from both states, and *similarity* among states can thus be measured. Algorithmic verification complexities are further studied for MDP and turn-based games [32]. Metric-based approaches allow to analyze similarity with a quantitative measure, in which sense our approach is more strict. However our definition of PA-simulation is purely by actions and strategies, while metric-based approach is more target-based as it defines similarity on states by the ability to achieve same outcomes with similar probabilities.

More recently, algorithmic verification of turn-based and concurrent games have been implemented in an extension of PRISM [33, 11]. The properties can be specified as state formulas, path formulas and reward formulas. The verification procedure requires solving matrix games for concurrent game structures, and it applies value iteration algorithms to approach the goal (similar to [34, 31]). For unbounded properties, the synthesised strategy is memoryless (but only ϵ-optimal strategies). Finite-memory strategies are synthesised for bounded properties. The model checking algorithms for PAMu in PGS may be extended from the existing algorithms implemented for PRISM.

9. Conclusions and Future Work

In this paper, we have presented sound and complete modal characterisations of PA-simulation and PA-bisimulation for concurrent games by introducing a new modal logic $\mathcal{L}^{\oplus}$ with its sub-logic $\mathcal{L}^{\ominus}$ and its extension PAMu (with fixpoints). All three logics incorporate nondeterministic and probabilisitic features and express the ability of the players to enforce a property in the current state of a probabilistic game structure (PGS). In the future, we aim to study proof systems for $\mathcal{L}^{\oplus}$ and PAMu.

Since mixed actions can always be encoded as a linear combination of "normal" (deterministic) actions, the model checking problem of the next step strategy modality $\langle\langle\mathcal{I}\rangle\rangle\varphi$ can be effectively reduced to Linear Programming (LP). Since both players have complete view over states in a PGS, the model checking problem for $\mathcal{L}^{\ominus}$ with finite conjunction/disjunction (also extended with fixpoint operators) is likely to be decidable in P. However, it seems challenging to deal with formulas that contain the negation operator on a probabilistic summation. We plan to take a close look at the verification problem of $\mathcal{L}^{\mu}$ (or a fragment of $\mathcal{L}^{\mu}$) in the future.

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