A Characterisation of Open Bisimilarity using an Intuitionistic Modal Logic

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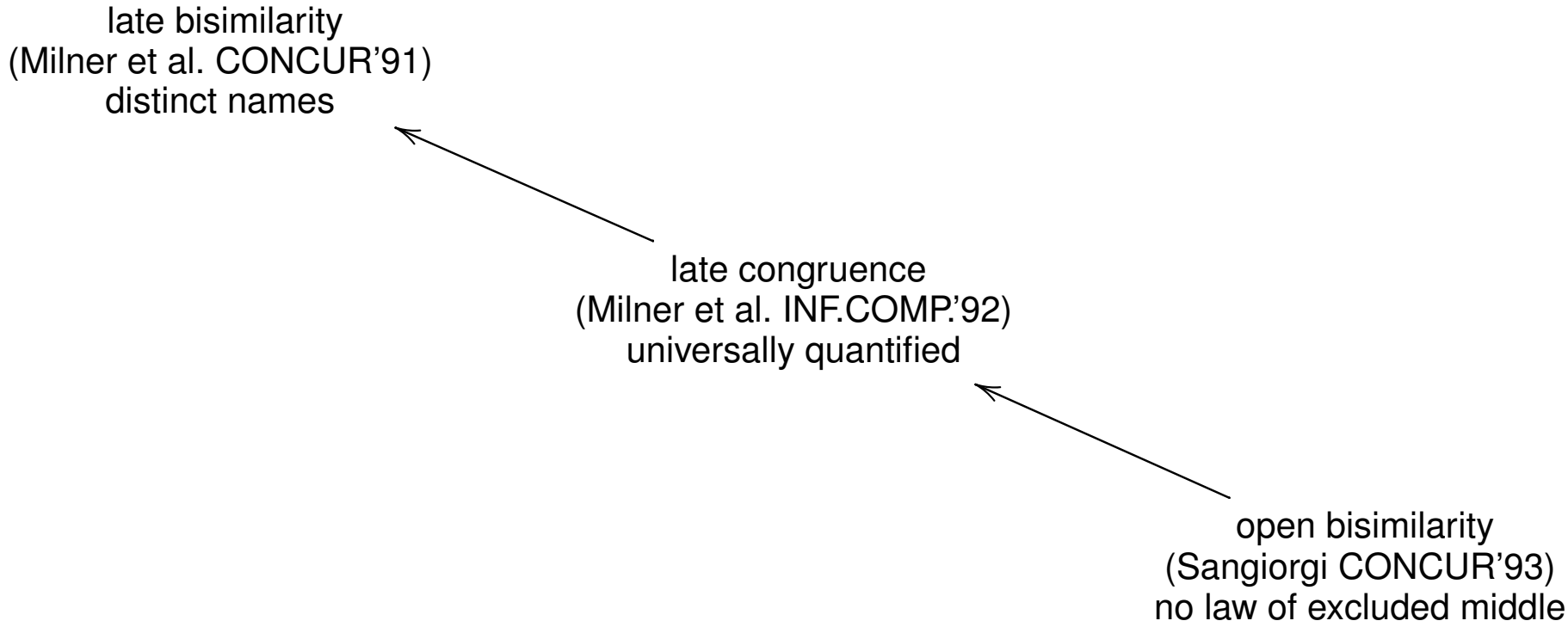
All you need to know about open bisimilarity



- No distinct name assumption (names are variables).
- Closed by substitutions at every step (variables may be unified later).
- Is automatically a congruence (preserved under input prefixes).
- Still a bisimulation: so robust under unanticipated change of context during runtime.
- Permits a lazy call-by-need instantiation of inputs, which is efficient to automate.

Why no law of excluded middle?

Classically, $\phi \vee \neg\phi$ is a tautology and $\left[\pi\right]\phi = \neg\left\langle\pi\right\rangle\neg\phi$.
Hence for all classical modal logics:

$$\overline{a}b \parallel c(x) \models \left\langle\tau\right\rangle\mathtt{tt} \vee \left[\tau\right]\mathtt{ff}$$

Intuitionistically, "Reachable worlds" are processes accessible by substitutions. E.g.

$$\overline{a}b \parallel c(x) \leq (\overline{a}b \parallel c(x))\{{}^{a}/_{c}\} \qquad \text{and} \qquad \overline{a}b \parallel a(x) \xrightarrow{\tau} 0$$

Closing modalities under all reachable worlds (**intuitionistic heredity**):

$$\overline{a}b \parallel c(x) \not\models \left\langle\tau\right\rangle\mathtt{tt} \vee \left[\tau\right]\mathtt{ff}$$

$\overline{a}b \parallel c(x) \not\models \left\langle\tau\right\rangle\mathtt{tt}$, — no τ-transition in world where a and c are not equal.

$\overline{a}b \parallel c(x) \not\models \left[\tau\right]\mathtt{ff}$, — some τ-transition in world where $a = c$.

For congruences, quantify over all names

For congruences (open bisimilarity and late congruence):

$$[x = y]\tau \qquad \not\sim \qquad 0$$

Distinguishing strategy, in world where $x = y$:

$$([x = y]\tau)\{^{y}/_{x}\} \xrightarrow{\tau} 0 \qquad \text{but} \qquad 0\{^{y}/_{x}\} \not\downarrow_{\tau}$$

Distinguishing formula biased to the left:

$$[x = y]\tau \models \big[x = y\big]\big\langle\tau\big\rangle\mathtt{tt} \qquad \text{and} \qquad 0 \not\models \big[x = y\big]\big\langle\tau\big\rangle\mathtt{tt}$$

Distinguishing formula biased to the right:

$$[x = y]\tau \not\models \big[\tau\big]\mathtt{ff} \qquad \text{and} \qquad 0 \models \big[\tau\big]\mathtt{ff}$$

No duality between left and right formulae

All bisimulations agree:

$$[x = y]\tau \quad \not\sim \quad \tau$$

Distinguishing formula biased to the right:

$$[x = y]\tau \not\models \langle\tau\rangle\mathrm{tt} \qquad \text{and} \qquad 0 \models \langle\tau\rangle\mathrm{tt}$$

Dual formula is **not distinguishing**.

$$[x = y]\tau \not\models \neg\langle\tau\rangle\mathrm{tt} \qquad \text{and} \qquad 0 \not\models \neg\langle\tau\rangle\mathrm{tt}$$

[read as “there is no reachable world in which we can do a τ-transition”]

What is a distinguishing formula biased to the left?

No duality between left and right formulae

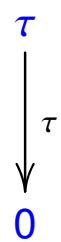
All bisimulations agree:

$$[x = y]\tau \quad \not\sim \quad \tau$$

What is a distinguishing formula biased to the left?

Distinguishing strategy (right process leads):

$$[x = y]\tau$$



Since right leads, for formula biased to the left, write **box**:

$$[\tau](\ldots\ldots\ldots\ldots)$$

No duality between left and right formulae

All bisimulations agree:

$$[x = y]\tau \quad \not\sim \quad \tau$$

What is a distinguishing formula biased to the left?

Distinguishing strategy (right process leads):

$[x = y]\tau$ $\xrightarrow{\tau}$ only in worlds where $x = y$

τ $\xrightarrow{\tau}$ 0

Since right leads and left only follows in worlds where $x = y$, write $x = y$ as post-condition:

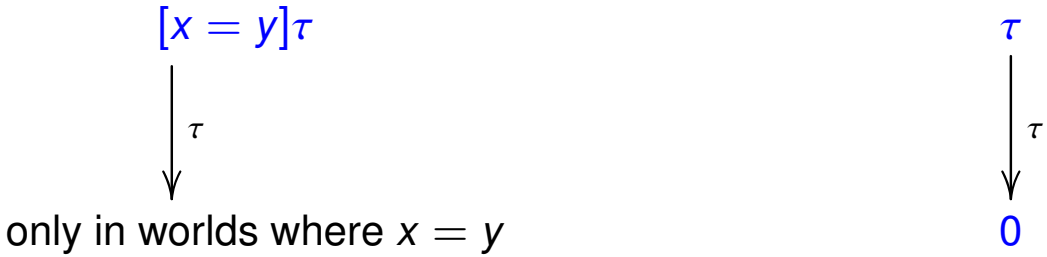
$$\big[\tau\big]\big\langle x = y\big\rangle \mathrm{tt}$$

No duality between left and right formulae

All bisimulations agree:

$$[x = y]\tau \quad \not\sim \quad \tau$$

Distinguishing strategy (right process leads):



Distinguishing formula biased to the left:

$$[x = y]\tau \models \big[\tau\big]\big\langle x = y\big\rangle \mathrm{tt} \qquad \text{and} \qquad 0 \not\models \big[\tau\big]\big\langle x = y\big\rangle \mathrm{tt}$$

Generating distinguishing formulae algorithmically

Not open bisimilar:

$$\tau + \tau.\tau + \tau.[x = y]\tau \quad \not\sim \quad \tau + \tau.\tau$$

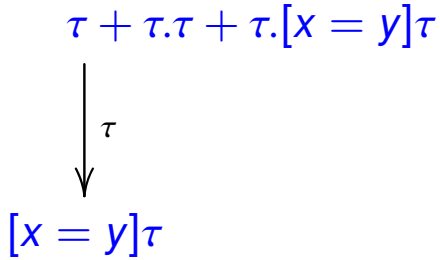
Distinguishing strategy (left process leads first):

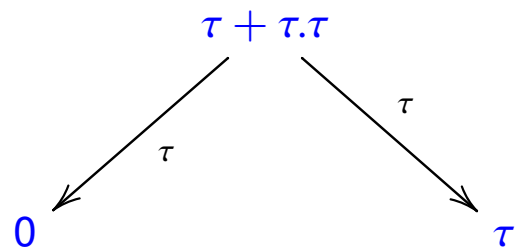
We just saw:

$$[x = y]\tau \not\sim 0 \qquad \text{and} \qquad [x = y]\tau \not\sim \tau$$

Generating distinguishing formulae algorithmically

Distinguishing strategy (left process leads first):





We just saw:

$$[x = y]\tau \models \big[x = y\big]\big\langle\tau\big\rangle\mathtt{tt} \qquad 0 \models \big[\tau\big]\mathtt{ff}$$

$$[x = y]\tau \models \big[\tau\big]\big\langle x = y\big\rangle\mathtt{tt} \qquad \tau \models \big\langle\tau\big\rangle\mathtt{tt}$$

Since left process leads: diamond on left, box on right.

$$\big\langle\tau\big\rangle(\ldots\ldots\ldots\ldots\ldots\ldots) \qquad \big[\tau\big](\ldots\ldots\ldots\ldots\ldots\ldots)$$

Generating distinguishing formulae algorithmically

Distinguishing strategy (left process leads first):

We just saw:

$$[x = y]\tau \models \big[x = y\big]\big\langle\tau\big\rangle\mathtt{tt} \qquad 0 \models \big[\tau\big]\mathtt{ff}$$

$$[x = y]\tau \models \big[\tau\big]\big\langle x = y\big\rangle\mathtt{tt} \qquad \tau \models \big\langle\tau\big\rangle\mathtt{tt}$$

Since left process leads: conjunction on left, disjunction on right.

$$\big\langle\tau\big\rangle\big(\big[x = y\big]\big\langle\tau\big\rangle\mathtt{tt} \wedge \big[\tau\big]\big\langle x = y\big\rangle\mathtt{tt}\big) \qquad \big[\tau\big]\big(\big[\tau\big]\mathtt{ff} \vee \big\langle\tau\big\rangle\mathtt{tt}\big)$$

Generating distinguishing formulae algorithmically

Not open bisimilar:

$$\tau + \tau.\tau + \tau.[x = y]\tau \qquad \not\sim \qquad \tau + \tau.\tau$$

Distinguishing formula biased to left.

$$\tau + \tau.\tau + \tau.[x = y]\tau \models \langle\tau\rangle\big([x = y]\langle\tau\rangle\mathrm{tt} \wedge [\tau]\langle x = y\rangle\mathrm{tt}\big)$$

Distinguishing formula biased to right.

$$\tau + \tau.\tau \models [\tau]\big([\tau]\mathrm{ff} \vee \langle\tau\rangle\mathrm{tt}\big)$$

Only distinguishing if we drop law of excluded middle for name equality. I.e.

$$x = y \text{ or } x \neq y \qquad \text{iff} \qquad \tau + \tau.\tau + \tau.[x = y]\tau \models [\tau]\big([\tau]\mathrm{ff} \vee \langle\tau\rangle\mathrm{tt}\big).$$

[Mechanically proven in intuitionistic proof assistant]

Results for modal logic $\mathcal{OM}$

Theorem (soundness)

If $P \sim Q$ then, for all formulae ϕ, $P \models \phi$ iff $Q \models \phi$.

Proof.

By induction on structure of ϕ. [mechanised] □

Theorem (soundness)

Whenever, for all formulae ϕ, $P \models \phi$ iff $Q \models \phi$, we have that $P \sim Q$.

Proof.

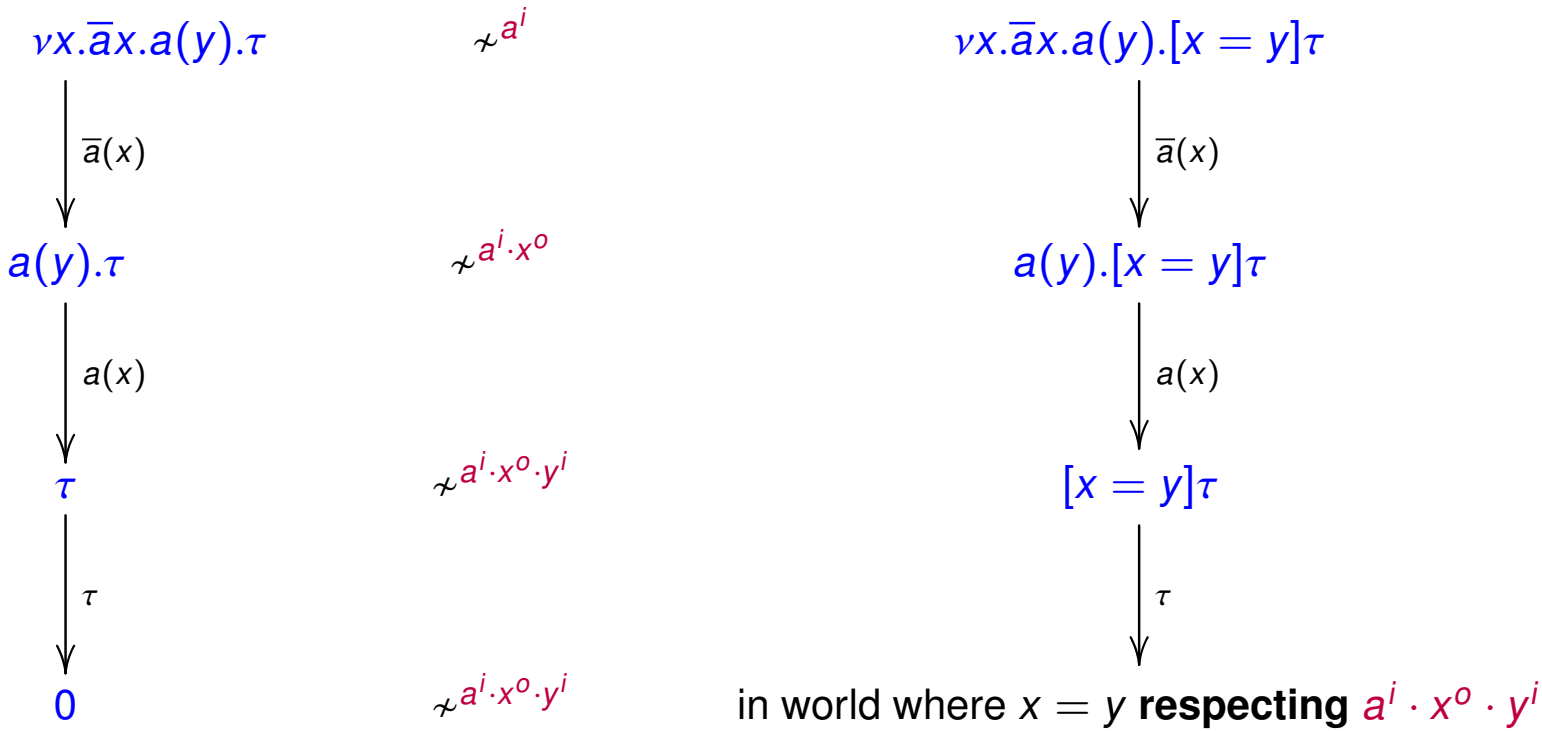
If $P \nsim Q$, by induction on depth of a distinguishing strategy, construct ϕ_L and ϕ_R such that:

- $P \models \phi_L$ and $Q \not\models \phi_L$;
- $P \not\models \phi_R$ and $Q \models \phi_R$.

Proof is constructive, so yields algorithm for generating distinguishing formulae. [implemented] □

Subtleties of $\mathcal{OM}$: respectful substitutions

Distinguishing strategy (left always leading):

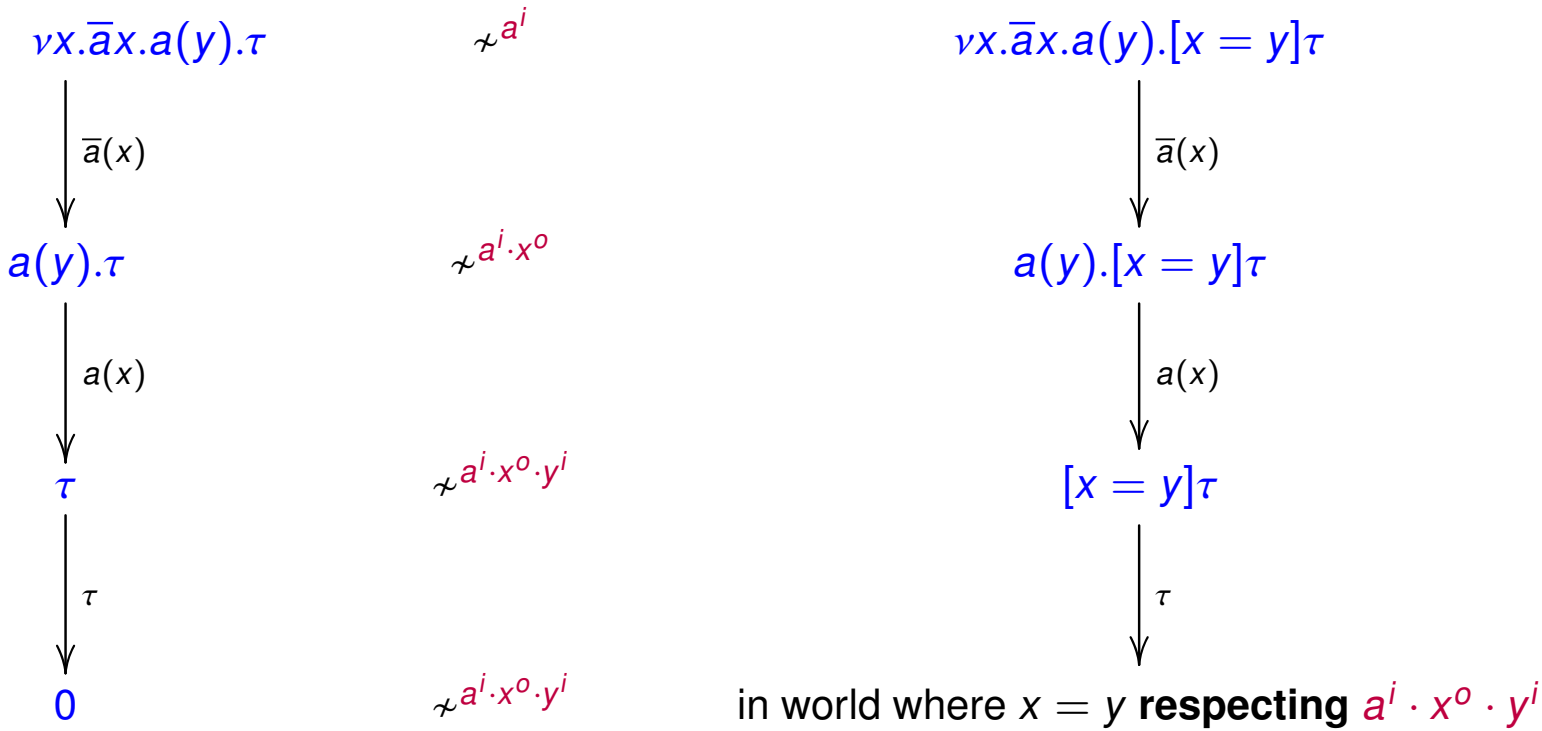


Distinguishing formulae (base case):

$$\tau \models^{a^i \cdot x^o \cdot y^i} \langle\tau\rangle \mathrm{tt} \quad \text{and} \quad [x = y]\tau \models^{a^i \cdot x^o \cdot y^i} [\tau]\langle x = y\rangle \mathrm{tt}$$

Subtleties of $\mathcal{OM}$: respectful substitutions

Distinguishing strategy (left always leading):

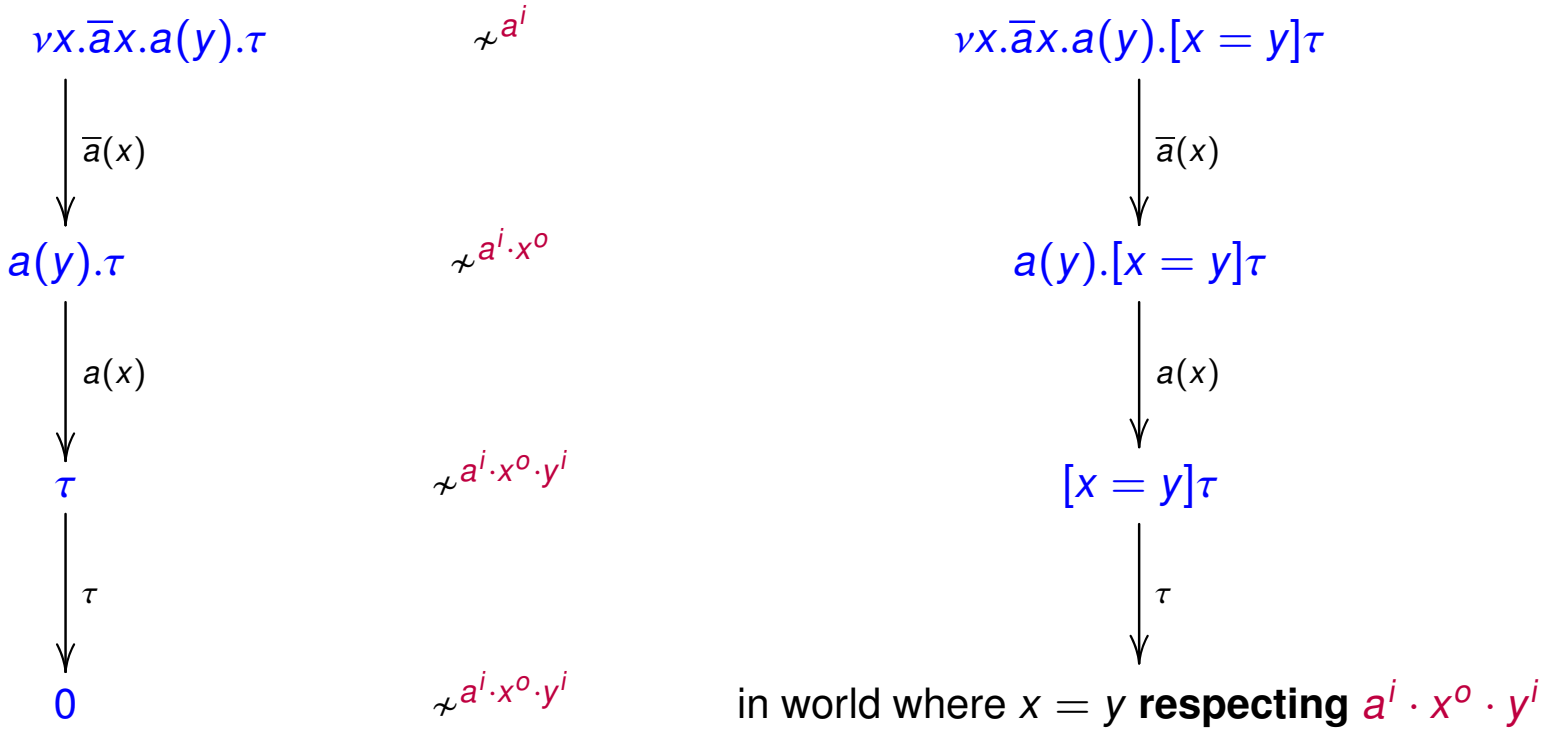


Distinguishing formulae (inductive case):

$$a(y).\tau \models^{a^i \cdot x^o} \langle a(y) \rangle \langle \tau \rangle \mathrm{tt} \quad \text{and} \quad a(y).[x=y]\tau \models^{a^i \cdot x^o} [a(y)][\tau]\langle x=y \rangle \mathrm{tt}$$

Subtleties of $\mathcal{OM}$: respectful substitutions

Distinguishing strategy (left always leading):



Distinguishing formulae:

$\nu x.\overline{a}x.a(y).\tau \models^{a^i} \langle\overline{a}(x)\rangle\langle a(y)\rangle\langle\tau\rangle\mathrm{tt}$ and $\nu x.\overline{a}x.a(y).[x=y]\tau \models^{a^i} [\overline{a}(x)][a(y)][\tau]\langle x=y\rangle\mathrm{tt}$

Subtleties of $\mathcal{OM}$: input modalities

Late modality for diamond:

$$P \models \langle a(z) \rangle \phi \qquad \text{iff} \qquad \exists\, Q.\ P \xrightarrow{a(z)} Q \text{ and } \forall z.\ Q \models \phi.$$

Basic modality for box:

$$P \models [a(z)] \phi \qquad \text{iff} \qquad \forall\, Q.\ P \xrightarrow{a(z)} Q \text{ implies } \forall z.\ Q \models \phi.$$

Modalities have independent interpretations!

In contrast, classical Late modality for box, is de Morgan dual to diamond:

$$P \models [a(x)]^{L} \phi \qquad \text{iff} \qquad \forall\, Q.\ P \xrightarrow{a(x)} Q \text{ implies } \exists x.\ Q \models \phi$$

Subtleties of $\mathcal{OM}$: input modalities

Neither open bisimilar nor late bisimilar:

$$a(x).\tau + a(x) + a(x).[x = a]\tau \quad \not\sim \quad a(x).\tau + a(x)$$

Distinguishing formulae biased to the right:

$$a(x).\tau + a(x) \models \big[a(x)\big]\big(\big\langle\tau\big\rangle\mathtt{tt} \vee \big[\tau\big]\mathtt{ff}\big)$$

However, fails to be distinguishing, both:

- classically, with law of excluded middle;
- intuitionistically, with late box modality.

Intuitionistic with basic box modalities and late diamond modalities.

Conclusion

- $\mathcal{OM}$ is the first modal logic *proven* to **characterise open bisimilarity**.
- $\mathcal{OM}$ is fundamentally an intuitionistic modal logic!
  - **Intuitionistic hereditary**, given by *respectful* substitutions.
  - No **law of excluded middle** (in modal logic and meta-framework).
  - Modalities have **independent** interpretations . . . no **de Morgan dualities**.
- Satisfaction and generation of distinguishing formulae are **implemented**.
- Techniques **general** and known to extend to other calculi and bisimulations.
- Perspectives: open bisimilarity permits efficient **symbolic** decision procedures.
  - especially good for infinite inputs in *cryptographic calculi*.
  - What about *intuitionistic symbolic model checking* (invariant under open bisimilarity)?

A more challenging example!

These processes are late congruent to $\tau + \tau.\tau$.

$$\tau.(\tau + \tau.\tau + \tau.[x = y][w = z]\tau)$$

$$\not\sim$$

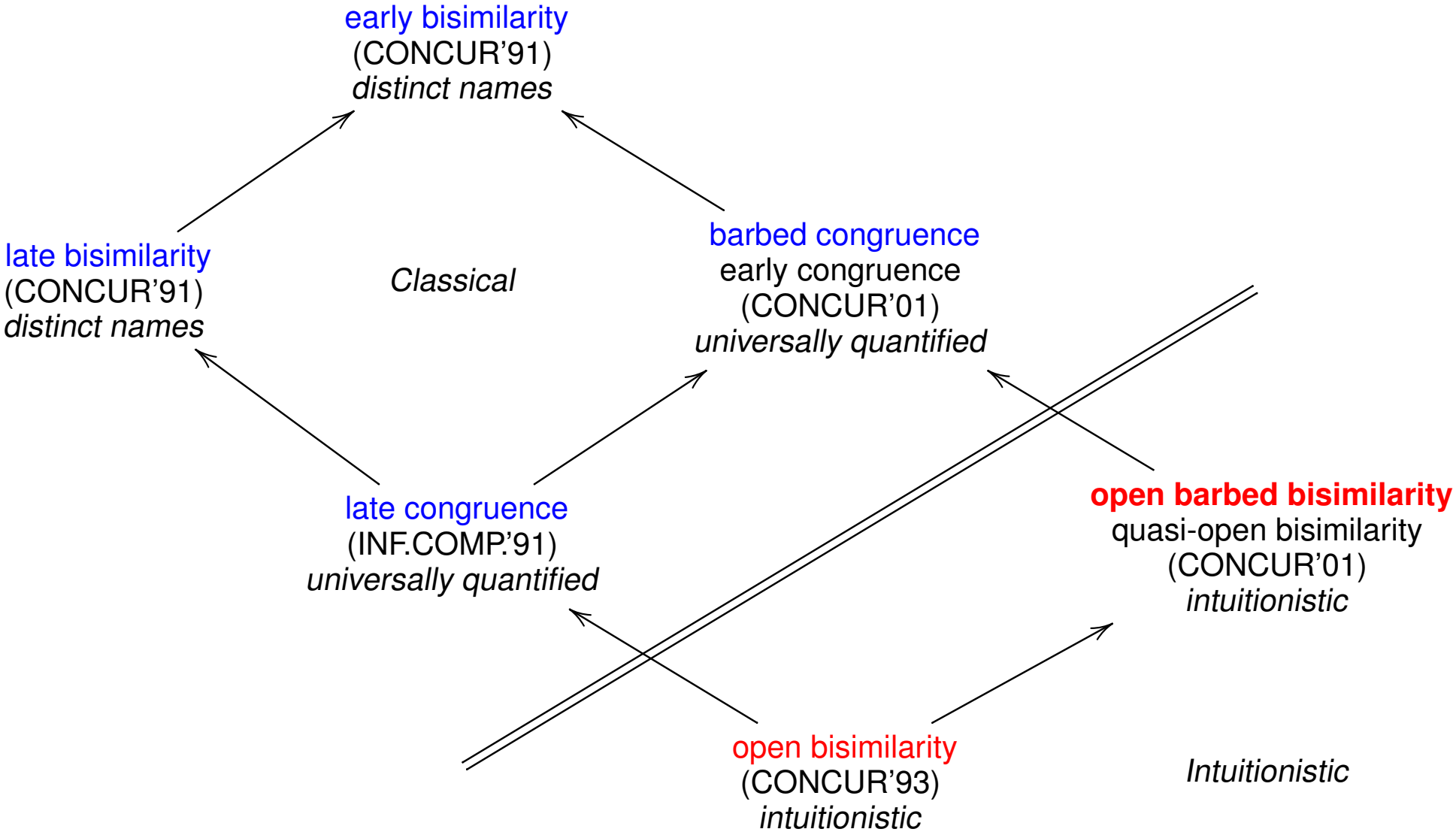
$$\tau.(\tau + \tau.\tau + \tau.[x = y]\tau) + \tau.(\tau + \tau.\tau + \tau.[x = y][w = z]\tau)$$

They are not open bisimilar!

What are the distinguishing strategies?

What are the distinguishing formulae?

Results work for early transition systems



Additional criterion for intuitionistic modal logics:

> *If law of excluded middle is forced we obtain a meaningful classical modal logic (open barbed bisimilarity becomes barbed congruence).*