Why men (and octopuses) cannot juggle a four ball cascade

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Abstract

The most common juggling pattern is the three ball cascade. A four ball cascade seems to be infeasible, but why? We formalize the notion of a cascade and the decomposition of juggling patterns in order to give an answer to this question.

1 Introduction

It is not difficult to learn how to juggle. One only needs a few hours of practice and some perseverance in order to master the most common pattern, the three ball cascade (see Figure 1). This cascade is the most symmetric and regular pattern that can be thrown with three balls. The two hands move in alternation and the balls are also thrown in a fixed alternating order.

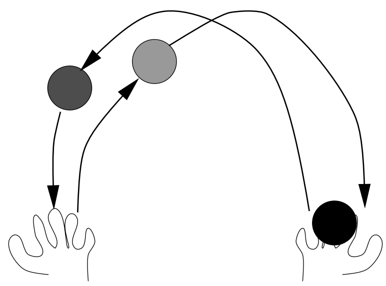

Figure 1: A three ball cascade

After having mastered a three ball cascade, the novice juggler asks himself the question how to juggle four balls. One would think that juggling four balls is qualitatively the same as juggling three balls, be it with harder constraints on timing and accuracy. But however far a person's physical skills may progress, a four ball cascade does not seem to be possible. One will come up with less symmetric patterns, such as the *shower* (see Figure 1) or with compound patterns such as *two columns* (see Figure 1).

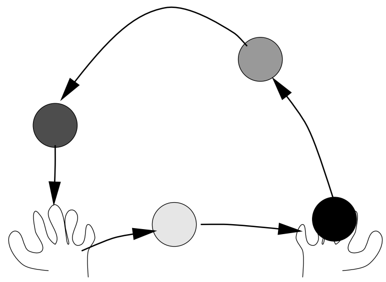

Figure 2: A shower

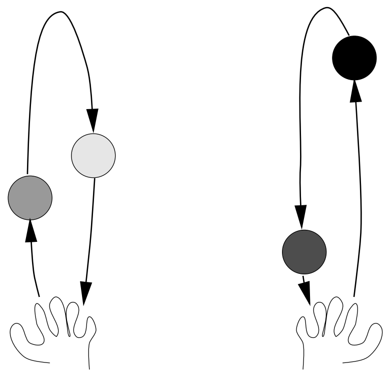

Figure 3: Two columns

Juggling a four ball cascade with two hands seems to be impossible, but why? It is certainly not due to a lack of skills, because experienced jugglers have the same problem. Experience shows that it is possible to juggle a cascade with every odd number of balls (up to 11, which is the current world record), but a cascade with an even number of balls remains impossible. There is probably a more fundamental reason. In this paper we will formulate an answer.

Thereto, we will give a formal definition of the notion of a cascade and using simple number theory, we will show that two hands cannot throw a four ball cascade.

Because we do not fix the number of hands in our definitions, our result will be more general. The main theorem that we will derive is the following. It is only possible to juggle a cascade if the number of hands and the number of balls are relatively prime, i.e. have no common divisors larger than 1. This theorem has already been stated (without a proof) as a proposition accompanying the PH.D-thesis of one of the authors (see [1]).

The extension to more than two hands is not purely artificial; there exist advanced juggling patterns involving more than one juggler.

In the remainder of this paper we first define a notation for juggling patterns and the notion of a cascade. By introducing partitions of a juggling pattern, we can make a distinction between a real cascade and a compound cascade.

Thanks are due to all members of the *Department of Pure Juggling* in the Watergraafsmeer (Amsterdam) for the fruitful discussions during the practical and theoretical juggling sessions, and to Tijn Borghuis whose comments helped much in improving this paper.

2 Notation

It has been recognized by the juggling community that some notation for communicating juggling patterns is needed. A variety of notations are used, ranging from completely informal cartoons to a compact and formal notation such as *site swaps* (see [2, 3]).

Since we intend to use mathematical methods for analyzing juggling patterns, we need a formal notation. Unfortunately, none of the existing notations used lines up neatly with our purposes. Therefore, we introduce a simple and ad-hoc notation for expressing juggling patterns.

In order to capture the essence of a juggling pattern, we have to abstract away physical properties, such as size and speed of the balls, direction of the throws, etc. The only essential information is the order in which the hands throw the balls. Thus we will describe a juggling pattern as a sequence of ball/hand-combinations.

Definition 1 (juggling patterns) A juggling pattern J is a (possibly infinite) sequence of pairs $\binom{h_1}{b_1}\binom{h_2}{b_2}\binom{h_3}{b_3}\ldots$, where the h_i are chosen from some finite set of hands H and the b_i from some finite set of balls B, such that each $h \in H$ and each $b \in B$ is part of a pair in J. This sequence means that first ball b_1 is thrown from hand h_1, then b_2 from h_2, etc.

For example, the 3-ball cascade in Figures 1 is the pattern $\binom{L}{1}\binom{R}{2}\binom{L}{3}\binom{R}{1}\binom{L}{2}\binom{R}{3}\binom{L}{1}\binom{R}{2}\binom{L}{3}\binom{R}{1}\binom{L}{2}\binom{R}{3}\ldots$, where the dots stand for a repetition of the shown order of pairs (using the hands L and R for obvious reasons).

Several observations can be made about this notation. First, we did not include the catching of a ball as a separate event. Although the timing of the catches influences the taste of the pattern, we do not think it is essential to the pattern. As a consequence, we are not able to express the difference between *hot potato juggling*, in which the balls stay as short as possible in the hands, and *lazy juggling*, which is the opposite. The only requirement that is present is the practical requirement that a ball is caught somewhere between two successive throwings of that ball. Secondly, we cannot describe patterns with synchronous throws, which is the case if two throws occur at the same time. Neither is the notation suited to express multiplexing, which means that at certain times more than one ball is in the same hand. Thirdly, we did not include a possibility for explicitly stating that some pattern is executed repeatedly.

However, for many basic patterns, and certainly for a cascade, this notation is all we need.

If $J = \binom{h_1}{b_1}\binom{h_2}{b_2}\binom{h_3}{b_3}\ldots$, we call $h_1 h_2 h_3 \ldots$ its hand pattern and $b_1 b_2 b_3 \ldots$ its ball pattern. For example, the 3-ball cascade $\binom{L}{1}\binom{R}{2}\binom{L}{3}\binom{R}{1}\binom{L}{2}\binom{R}{3}$ $\binom{L}{1}\binom{R}{2}\binom{L}{3}\binom{R}{1}\binom{L}{2}\binom{R}{3}\ldots$ has hand pattern $LRLRLRLR\ldots$ and ball pattern $123123123\ldots$.

In practice only periodic juggling patterns are really interesting. This periodicity can easily be found back in the hand and ball patterns, as can be seen in the Theorem below:

Theorem 2 (periodicity of juggling patterns) Let a juggling pattern J be such that its hand pattern is periodic with period p_H and its ball pattern is periodic with period p_B. Then the pattern itself is periodic with period $\mathrm{scm}(p_H, p_B)$ (where scm stands for smallest common multiple). Furthermore, if the periods p_H and p_B are both prime periods (that is, the patterns are not periodic with any smaller period), then $\mathrm{scm}(p_H, p_B)$ is the prime period of J.

Proof This follows from the following basic observation. The juggling pattern is periodic with some period p if and only if both the hand and the ball pattern are periodic with that same period p. ■

3 The cascade

As stated in the introduction, a cascade is a pattern in which the hands throw alternately and the balls are thrown in a fixed order. This observation leads to the following definition.

Definition 3 (cascade) A cascade is a juggling pattern in which both the hand pattern and the ball pattern are cyclic (that is, the hand pattern is periodic and each hand shows up exactly once in each prime period, and likewise for the ball pattern).

Under this definition there is exactly one cascade for each given number of hands and balls. But if we look at the 2-hand, 4-ball cascade, we see the following juggling pattern:

$\binom{L}{1}\binom{R}{2}\binom{L}{3}\binom{R}{4}\binom{L}{1}\binom{R}{2}\binom{L}{3}\binom{R}{4}\ldots$

Looking at the pattern closely, one notices that it is in fact a superposition of two separate patterns: the L-hand is throwing and catching balls 1 and 3, and the R-hand is throwing and catching balls 2 and 4, without any interaction between the two. The 2-hand, 4-ball cascade is just two 1-hand, 2-ball cascades in parallel (see the *two columns* pattern from Figure 1). Because of this, it does not count as a real cascade. So now the question becomes: Which cascades (and other juggling patterns) are real in the sense that they are not composed from simpler patterns, and which are not? To answer this question, we must first formally define what a real juggling pattern is. For reasons which will soon become clear, we will call these prime patterns.

4 Closed partitions and prime patterns

For a juggling pattern to be real, we want it to be impossible to divide its events into two or more groups, in such a way that these groups are not interconnected by sharing a hand or ball.

Definition 4 (closed partition) Let $J = \binom{h_1}{b_1}\binom{h_2}{b_2}\ldots$ be a juggling pattern. A closed partition of J is a set $J_1, \ldots, J_n$ ($n \in \mathbb{N}$) of subsequences of J, such that (let $J_i = \binom{h_{n_{i1}}}{b_{n_{i1}}}\binom{h_{n_{i2}}}{b_{n_{i2}}}\ldots$):

1. The J_i form a partition of J, that is, each J_i is nonempty, and for each $k \in N$ there is exactly one i and one j such that $n_{ij} = k$.

2. If either $h_i = h_j$ or $b_i = b_j$, then the pairs $\binom{h_i}{b_i}$ and $\binom{h_j}{b_j}$ are in the same subsequence.

A juggling pattern is called *prime* if and only if its only closed partition is the trivial partition, i.e. $n = 1$ and $J_1 = J$.

Now we have the material to distinguish the 'real' cascades from the 'false' ones. A juggling pattern is 'real' if and only if it is prime. For example, the 4-ball cascade $\binom{L}{1}\binom{R}{2}\binom{L}{3}\binom{R}{4}\binom{L}{1}\binom{R}{2}\binom{L}{3}\binom{R}{4}\ldots$ is not prime, as it can be partitioned in the subsequences $\binom{L}{1}\binom{L}{3}\binom{L}{1}\binom{L}{3}\ldots$ and $\binom{R}{2}\binom{R}{4}\binom{R}{2}\binom{R}{4}\ldots$. On the other hand, the 3-ball cascade $\binom{L}{1}\binom{R}{2}\binom{L}{3}\binom{R}{1}\binom{L}{2}\binom{R}{3}\binom{L}{1}\binom{R}{2}\binom{L}{3}\binom{R}{1}\binom{L}{2}\binom{R}{3}\ldots$ is prime.

The term 'prime patterns' has been chosen because they can be used to build up all other patterns, as specified in the following theorem:

Theorem 5 Each juggling pattern J has a unique closed partition $J_1, \ldots, J_n$ such that each J_i is a prime pattern.

Proof If $J_1, \ldots, J_m$ and $J'_1, \ldots, J'_n$ are both closed partitions of J, then a new partition, which partitions all subsequences J_i and J'_i, can be obtained by looking at the subsequences $J_i \cap J'_j$, and discarding those that are empty. It is not hard to see that this will lead to a closed partition again. Taking such an intersection over all possible closed partitions will give the unique closed partition into prime patterns. As no closed partition can consist of more than $\min(p_H, p_B)$ different subsequences, their number is finite, so the intersection is finite as well. ■

Next, our aim is to find some necessary and/or sufficient condition on when a juggling pattern is prime. For this we will use the following basic number theoretic result:

Theorem 6 (Chinese remainder theorem) Let m, n, x and y be four numbers such that m and n are relatively prime. Then there exists a number z such that $z \equiv x \mod m$ and $z \equiv y \mod n$.

Proof Can be found in any book on number theory, see for example [4]. ■

For periodic patterns this leads to the following sufficient condition for a juggling pattern to be prime.

Theorem 7 Let J be a juggling pattern of which the hand pattern is periodic with period p_H and the ball pattern is periodic with period p_B, where p_H and p_B are relatively prime. Then J is a prime pattern.

Proof We will prove that for every pair $h \in H$ and $b \in B$ there is a pair $\begin{pmatrix} h \\ b \end{pmatrix}$ in the sequence J. From this it can easily be seen that J has no closed partition but the trivial one, and thus is prime.

Let h be any element of H, and let b be any element of B. Let J be $\begin{pmatrix} h_1 \\ b_1 \end{pmatrix} \begin{pmatrix} h_2 \\ b_2 \end{pmatrix} \ldots$, and let m and n be chosen such that $h_m = h$ and $b_n = b$. Now find a number k such that $k \equiv m \mod p_H$ and $k \equiv n \mod p_B$. That such a number exists, follows from the Chinese Remainder Theorem. Because $k \equiv m \mod p_H$ we have $h_k = h_m = h$, likewise $b_k = b$, so $\begin{pmatrix} h_k \\ b_k \end{pmatrix} = \begin{pmatrix} h \\ b \end{pmatrix}$. ■

5 Back to the cascade

In the last theorem we saw that all juggling patterns where the periods of hands and balls are relatively prime are prime patterns, and thus are real patterns, which cannot be subdivided into separate independent patterns. In general the reverse is not true. The simplest counterexample is the 2-ball shower, which can be found by removing the two dark balls in Figure 1. This has a juggling pattern, consisting of a constant repetition of the sequence $\begin{pmatrix} L \\ 1 \end{pmatrix}\begin{pmatrix} R \\ 1 \end{pmatrix}\begin{pmatrix} L \\ 2 \end{pmatrix}\begin{pmatrix} R \\ 2 \end{pmatrix}$. This is clearly a prime pattern, although the hand pattern has prime period 2 and the ball pattern has prime period 4, so these prime periods are not relatively prime. However, as we shall see in the next theorem, the reverse of Theorem 7 is true if we restrict ourselves to cascades only:

Theorem 8 Let J be a cascade with p_H hands and p_B balls. Then J can be partitioned in exactly $\gcd(p_H, p_B)$ prime patterns (where gcd stands for greatest common divisor). Furthermore, each of these prime patterns is a cascade with $\frac{p_H}{\gcd(p_H,p_B)}$ hands and $\frac{p_B}{\gcd(p_H,p_B)}$ balls.

Proof We will prove this by constructing the prime patterns J_i.

Let $J = \begin{pmatrix} h_1 \\ b_1 \end{pmatrix}\begin{pmatrix} h_2 \\ b_2 \end{pmatrix}\ldots$. We define J_i, for i ranging from 1 to $\gcd(p_H, p_B)$ by $\begin{pmatrix} h_k \\ b_k \end{pmatrix} \in J_i \Leftrightarrow i \equiv k \mod \gcd(p_H, p_B)$. We claim that this is a closed partition of J, and that each J_i is prime.

It is easy to see that this is indeed a partition of J. To see that it is a closed partition, let $h \in H$ be any hand, and let the numbers m and n be such that $h_n = h_m = h$. Then, as the hand pattern is cyclic with period p_H, and every hand occurs exactly once in each cycle, $n \equiv m \mod p_H$, and thus $n \equiv m \mod \gcd(p_H, p_B)$ as well. Therefore, $\begin{pmatrix} h_m \\ b_m \end{pmatrix}$ and $\begin{pmatrix} h_n \\ b_n \end{pmatrix}$ are in the same subsequence J_i. Of course, the proof for balls is analogous, from which it can be seen that this partition is indeed closed.

Next we prove that every pattern J_i is prime. To do so, take h and b such, that there are, for some h' and b', pairs $\begin{pmatrix} h \\ b' \end{pmatrix}$ and $\begin{pmatrix} h' \\ b \end{pmatrix}$ in some given J_i. We will prove that there is a pair $\begin{pmatrix} h \\ b \end{pmatrix}$ in J_i, from which we can conclude that J_i is prime.

As there are pairs $\begin{pmatrix} h \\ b' \end{pmatrix}$ and $\begin{pmatrix} h' \\ b \end{pmatrix}$ in J_i, there must be some numbers m and n, such that $h_{m.\gcd(p_H,p_B)+i} = h$ and $b_{n.\gcd(p_H,p_B)+i} = b$. By definition of gcd, $\frac{p_H}{\gcd(p_H,p_B)}$ and $\frac{p_B}{\gcd(p_H,p_B)}$ are relatively prime, and because $m \equiv n \equiv i \mod \gcd(p_H, p_B)$, $m - i$ and $n - i$ are both divisible by $\gcd(p_H, p_B)$. Thus we can use the Chinese Remainder Theorem to find an l such that $l \equiv \frac{m-i}{\gcd(p_H,p_B)} \mod \frac{p_H}{\gcd(p_H,p_B)}$ and $l \equiv \frac{n-i}{\gcd(p_H,p_B)} \mod \frac{p_B}{\gcd(p_H,p_B)}$. From this we can conclude that $l.\gcd(p_H, p_B) \equiv m - i \mod p_H$, so $l.\gcd(p_H, p_B) + i \equiv m \mod p_H$, so $h_{l.\gcd(p_H,p_B)+i} = h_m = h$. Likewise $b_{l.\gcd(p_H,p_B)+i} = b$, so there is a pair $\begin{pmatrix} h \\ b \end{pmatrix}$ in J_i. As this holds for each h and b, we can conclude that J_i is prime.

Let the hand pattern going with the juggling pattern J_i be denoted by H_i, and its ball pattern by B_i. It can easily be seen that H_i is a restriction of the hand pattern H to a subset of hands. As such, it can be seen to be cyclic as well. Furthermore, each H_i has an equal part of the set of hands, and as there are $\gcd(p_H, p_B)$ such subsequences, dividing p_H in equal parts, each such H_i contains exactly $\frac{p_H}{\gcd(p_H,p_B)}$ hands. Analogously, the ball pattern B_i is cyclic with $\frac{p_B}{\gcd(p_H,p_B)}$ balls. Thus, the patterns J_i are cascade patterns with $\frac{p_H}{\gcd(p_H,p_B)}$ balls and $\frac{p_B}{\gcd(p_H,p_B)}$ balls. ■

Concluding, we get to our main result:

Corollary 9 A true (that is, prime) cascade with h hands and b balls exists if and only if h and b are relatively prime.

6 Conclusions

The purpose of this paper was to use mathematical techniques to get insight in the properties of juggling patterns. We studied the most common pattern, the cascade. Our definitions are based on the assumption that a cascade is a pattern with maximal symmetry. Of course, since the notion of a cascade has never been defined unambiguously before, there cannot be a proof that our definition is correct. However, most of the jugglers that we have discussed this definition with seem to agree that this is a reasonable definition. Nevertheless, it is not to every juggler evident how to generalize the notion of a cascade to more than two hands. Our definition can be used in such a way that for every number of hands and for every number of balls a unique juggling pattern can be derived which is fully symmetric (but possibly not prime) by simply letting the balls cycle through the hands in a fixed order.

Furthermore, we defined the notion of a composed pattern. The smallest patterns that cannot be composed into even smaller patterns are called prime patterns. Using simple mathematical reasoning and elementary number theory we obtained our main result. It is considered common sense among jugglers that one is only able to throw a cascade if the number of balls is odd. We have proven this observation formally and have also derived its generalizations that the number of hands and the number of balls should be relatively prime.

Our results are based on a very simple model of juggling patterns. The only structure needed for deriving our results is the order in which the hands throw the balls. We have abstracted away from all other properties of a juggling pattern.

An interesting question is whether there are other common juggling patterns, such as a shower or typical club passing patterns which can be characterized in the same easy way as the cascade. Having a collection of formal definitions of common patterns would help to classify juggling tricks and study their properties.

In this paper one such property has been defined, namely the primality of a juggling pattern, which means that the pattern is not composed of simpler patterns. Other interesting properties that might lend themselves for formalization might for example be symmetry (when is a pattern symmetric?) and forced moves (some or all hands are sometimes or always forced to throw a ball because the ball to be caught next is already airborne).

An interesting property of the class of cascade patterns is that it is closed under taking the dual of a pattern. By this we mean that we change the roles of the carrier sets for hands and balls, thus considering the balls as if they were the throwing actors and the hands as if they are being thrown. From a mathematical point of view the dual of a pattern is not much different from the pattern itself, but in practice the dual of a pattern can be some orders of magnitude harder (or easier) to master. Likewise, the aesthetical appreciation may also be very different. Consider for example the difference between a very difficult cascade with one hand and five balls and its dual, the trivial cascade with five hands and one ball. Such a duality might be an interesting subject in further juggling research.

In our definitions we made use of an ad-hoc notation for juggling patterns. For several reasons as explained before, we did not use the more generally accepted *site swap* notation, as has for example been done by [3] and [5]. The relation between our notation (and our results) and the site swap notation is worth investigating. Which class of site swap patterns is considered a cascade? Does this class coincide with our notion of cascade? How can one define the primality of a pattern in site swap notation? What is the most convenient extension of the site swap notation for denoting patterns with more than two hands?

We can think of several extensions of our notation in order to be able to use it for a larger class of juggling patterns. We mention synchronized throws (two or more throws exactly at the same time) and multiplexing of balls (two or more balls at the same moment in the same hand).

Our notation could quite easily be changed to get closer to the normal site swap notation. In the siteswap the left and right hands always interchange, while a series of numbers gives the amount of time that passes until the same ball is caught again. Our notation could readily be changed into a kind of 'double site swap'-notation, in which two sets of numbers give the delay until the next throw of the same ball respectively the next throw with the same hand. Compared to the 'normal' site swap, this notation is more cumbersome for simple patterns, but easier for patterns with irregular hand patterns, and more easily adapted to patterns with more than two hands.

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