Analogies and discrepancies between the vertex cover number and the weakly connected domination number of a graph

M. Lemańska[a], J. A. Rodríguez-Velázquez[b], Rolando Trujillo-Rasua[c,*]

[a]*Department of Technical Physics and Applied Mathematics*
Gdansk University of Technology, Poland
[b]*Departament d'Enginyeria Informàtica i Matemàtiques*
Universitat Rovira i Virgili, Spain
[c]*Interdisciplinary Centre for Security, Reliability and Trust*
University of Luxembourg

Abstract

A vertex cover of a graph $G = (V, E)$ is a set $X \subset V$ such that each edge of G is incident to at least one vertex of X. The vertex cover number $\tau(G)$ is the size of a minimum vertex cover of G. A dominating set $D \subseteq V$ is a weakly connected dominating set of G if the subgraph $G[D]_w = (N[D], E_w)$ weakly induced by D, is connected, where E_w is the set of all edges having at least one vertex in D. The weakly connected domination number $\gamma_w(G)$ of G is the minimum cardinality among all weakly connected dominating sets of G. In this article we characterize the graphs where $\gamma_w(G) = \tau(G)$. In particular, we focus our attention on bipartite graphs, regular graphs, unicyclic graphs, block graphs and corona graphs.

Keywords: Vertex cover number, weakly connected domination number

1. Introduction

Throughout this paper $G = (V, E)$ will be a finite, undirected, simple graph of order n. A *vertex cover* of G is a set $X \subset V$ such that each edge of G is incident to at least one vertex of X. A minimum vertex cover is a vertex cover of smallest possible cardinality. The *vertex cover number* $\tau(G)$ is the cardinality of a minimum vertex cover of G. A vertex cover of cardinality $\tau(G)$ is called a $\tau(G)$-set. The minimum vertex cover problem arises in various important applications, including in multiple sequence alignments in computational biochemistry (see for example [6]). In computational biochemistry there are many situations where conflicts between sequences in a sample can be resolved by excluding some of the sequences. Of course, exactly what constitutes a conflict must be precisely defined in the biochemical context. It is possible to define a conflict graph where the vertices represent the sequences in the sample and there is an

*Corresponding author. Phone: +352 466 644 5458. Fax: 466 644 3 5458. Email: rolando.trujillo@uni.lu

edge between two vertices if and only if there is a conflict between the corresponding sequences. The aim is to remove the fewest possible sequences that will eliminate all conflicts, which is equivalent to finding a minimum vertex cover in the conflict graph. Several approaches, such as the use of a parameterized algorithm [3] and the use of a simulated annealing algorithm [11], have been developed to deal with this problem.

A set $D \subseteq V$ is *dominating* in $G = (V, E)$ if every vertex of $V - D$ has at least one neighbor in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets in G.

The *neighborhood* of a vertex $v \in V$ is the set $N(v)$ of all vertices adjacent to v in G. For a set $X \subseteq V$, the *open neighborhood*, $N(X)$, is defined to be $\cup_{v \in X} N(v)$ and *the closed neighborhood* of X is defined as $N[X] = N(X) \cup X$. Then the degree of a vertex $v \in V$ is $deg(v) = |N(v)|$. Given a vertex v of $G = (V, E)$ and a set $X \subset V$, let $N_X(v) = \{u \in X : uv \in E\}$.

Recall that a graph G is (δ_1, δ_2)-*semiregular* if all its vertices have degree either δ_1 or δ_2. In a (δ_1, δ_2)-semiregular bipartite graph $G = (U \cup W, E)$ every vertex of U has degree δ_1 and every vertex of W has degree δ_2.

A dominating set $D \subseteq V$ is a *weakly connected dominating set* of G if the subgraph $G[D]_w = (N[D], E_w)$ weakly induced by D, is connected, where E_w is the set of all edges having at least one vertex in D. Dunbar et al. [2] defined the *weakly connected domination number* $\gamma_w(G)$ of a graph G to be the minimum cardinality among all weakly connected dominating sets of G. A weakly connected dominating set of cardinality $\gamma_w(G)$ is called a $\gamma_w(G)$-set.

The motivation of studying weakly connected dominating sets comes from the study of ad hoc wireless networks [1]. A crucial way in which these differ from current cellular networks is that they do not have a separate routing infrastructure such as a system of base-stations; the mobiles have to conduct their own communication through routing. In these networks it is necessary to set up the so-called backbone, *i.e.,* a set of vertices and the links between them that is in charge of routing. In the specialized literature there is a general consensus that the backbone should be a dominating set, *i.e.,* each vertex is either in the backbone or next to some vertex in it. Rajaraman in [7] said that the most basic clustering that has been studied in the context of ad hoc networks is based on dominating sets. Moreover, the following additional features are considered to be appealing: (a) the backbone should be "small" and (b) it should be connected or weakly connected. Computing small connected dominating sets has been the focus of many articles [8, 9, 10]. While connectivity appears to be a natural requirement, several authors have argued that the right notion to apply in the wireless context is weak connectivity [1].

The main goal of this article is the study of analogies and discrepancies between the vertex cover number and the weakly connected domination number of a graph. To begin with, we establish some preliminary results.

2. Preliminaries

Since every vertex cover is also a weakly connected dominating set, the following result holds.

Proposition 1. [2] *For any graph G of order n, $\gamma_w(G) \leq \tau(G)$.*

The following result will be useful in Section 4 where we show that for regular graphs of order n, $\gamma_w(G) = \tau(G)$ if and only if G is bipartite and $\gamma_w(G) = \frac{n}{2}$.

Theorem 2. [2] *For any connected graph G of order n, $\gamma_w(G) \leq \frac{n}{2}$.*

If T is a tree and D is a minimum weakly connected dominating set of T, then every edge of T has at least one of its vertices in D. So every weakly connected dominating set in a non-trivial tree T is also a vertex cover of T and every vertex cover of T is a weakly connected dominating set.

Proposition 3. [2] *For any tree T of order $n \geq 2$, $\tau(T) = \gamma_w(T)$.*

In general, there are some graphs for which these parameters are equal, but the concepts are not necessarily equivalent. For example, if we consider a cycle $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$, then $\gamma_w(C_6) = \tau(C_6) = 3$. The set $\{v_1, v_3, v_5\}$ is both a minimum weakly connected dominating set and a minimum vertex cover of C_6. But, for instance, the set $\{v_1, v_3, v_4\}$ is a minimum weakly connected dominating set, but it is not a vertex cover of C_6.

Of course, there are also graphs G for which $\gamma_w(G) < \tau(G)$. As a simple example, consider the cycle C_{2k+1} ($k \geq 1$), where $\gamma_w(C_{2k+1}) = k$ and $\tau(C_{2k+1}) = k + 1$.

Claim 4. *For the cycle graph C_n, $\gamma_w(C_n) = \tau(C_n)$ if an only if n is even.*

In order to show a nontrivial family of graphs where the difference between τ and γ_w is arbitrarily large, we establish the following bound on the weakly connected domination number of the hypercube graphs.

Proposition 5. *For every hypercube graph Q_k, $k \geq 2$,*

$$\gamma_w(Q_k) \leq 2^{k-2} + 1.$$

Proof. We know that the hypercube Q_k can be defined by recurrence as a Cartesian product graph, *i.e.*, $Q_k = Q_{k-1} \Box K_2$, $k \geq 1$, where $Q_1 = K_2$. Let $\{a, b\}$ be the set of vertices of K_2. $S_2 = \{(a, a), (b, b)\}$ is a weakly connected dominating set for Q_2 and $S_3 = \{(a, a, a), (b, b, a), (b, b, b)\}$ is a weakly connected dominating set for Q_3.

Now, let $S'_3 = \{(b, b, a)\}$. Note that $S''_3 = S_3 - S'_3$ is an independent dominating set for Q_3. So, in Q_4, $S_3 \times \{a\}$ is a weakly connected dominating set for the copy of Q_3 corresponding to a and $S''_3 \times \{b\}$ is a dominating set for the copy of Q_3 corresponding to b. Notice also that for every $x \in S_3''$, (x, a) and (x, b) are neighbors in Q_4. Thus, $S_4 = (S_3 \times \{a\}) \cup (S_3'' \times \{b\})$ is a weakly connected dominating set for Q_4.

Following this process, we take $S_5 = (S_4 \times \{a\}) \cup (S_4'' \times \{b\})$, where $S_4'' = S_4 - S_3' \times \{a\}$. As above, S_5 is a weakly connected dominating set for Q_5. In general, $S_k = (S_{k-1} \times \{a\}) \cup (S_{k-1}'' \times \{b\})$, is a weakly connected dominating set for Q_k, where $S_{k-1}'' = S_{k-1} - S_{k-2}' \times \{a\}$.

Moreover, $|S_k| = 2|S_{k-1}| - 1$, where $|S_2| = 2$ and $|S_3| = 3$. Hence, $|S_k| = 2^{k-2} + 1$. The proof is complete. □

It is well-known that for the hypercube Q_k, $\tau(Q_k) = 2^{k-1}$ (see, for instance, [4]). Thus, by Proposition 5 we have

$$\tau(Q_k) - \gamma_w(Q_k) \geq 2^{k-2} - 1.$$

So, the difference between $\tau(Q_k)$ and $\gamma_w(Q_k)$ is arbitrarily large.

The next sections are devoted to a characterization of those graphs G which satisfy $\gamma_w(G) = \tau(G)$. In particular, we focus our attention on bipartite graphs, regular graphs, unicyclic graphs, block graphs and corona graphs.

3. Bipartite graphs

Theorem 6. (König 1931, Egerváry 1931) *For bipartite graphs the size of a maximum matching equals the size of a minimum vertex cover.*

We will use König-Egerváry's theorem to prove the following result.

Theorem 7. *Let G be a Hamiltonian bipartite graph of order n. Then $\gamma_w(G) = \tau(G)$ if and only if G is isomorphic to the cycle graph C_n.*

Proof. Let $G = (U \cup W, E)$. We know that if G is Hamiltonian, then $|U| = |W|$. So, if $(v_1, v_2, ...v_n, v_1)$ is a Hamiltonian cycle of G, then we can take $U = \{v_1, v_3, ..., v_{n-1}\}$ and $W = \{v_2, v_4, ..., v_n\}$. Suppose there exists at least one vertex, say v_1, of degree greater than two. Then $S = \{v_1, v_4, v_6, ..., v_{n-2}\}$ is a dominating set and v_1 must be adjacent to at least one vertex belonging to $S - \{v_1\}$. Hence, $G[S]_w$ is connected and, as a consequence, $\gamma_w(G) \leq |S| = \frac{n}{2} - 1$. Since every bipartite Hamiltonian graph has a perfect matching, by König-Egerváry's theorem we conclude that $\tau(G) = \frac{n}{2}$. Therefore, if $\gamma_w(G) = \tau(G)$, then G has maximum degree $\delta \leq 2$ and, since G is Hamiltonian, we observe that G is isomorphic to the cycle graph C_n.

The converse is straightforward. □

Given a connected graph $G = (V, E)$, we say that $X \subset V$ is a cut set if the subgraph of G induced by $V - X$ is not connected.

Lemma 8. *Let $G = (U \cup W, E)$ be a bipartite graph where $|U| \leq |W|$. Let d be the minimum among the degrees of the vertices belonging to W. If $\gamma_w(G) = |U|$ and $d \geq 3$, then for every pair of vertices $u, v \in U$ such that $N(u) \cap N(v) \neq \emptyset$, it follows that $\{u, v\}$ is a cut set.*

Proof. Let $U = \{u_1, u_2, ..., u_r\}$ and $W = \{w_1, w_2, ..., w_t\}$. Suppose, without loss of generality, that u_1 and u_2 are adjacent to w_1 and $\{u_1, u_2\}$ is not a cut set. Let $S = \{w_1, u_3, u_4, ..., u_r\}$. Since $d \geq 3$, each vertex of $W - \{w_1\}$ must be adjacent to at least one vertex belonging to $S - \{w_1\}$. Hence, S is a dominating set.

Now, since $\{u_1, u_2\}$ is not a cut set, the subgraph of G obtained by removing u_1 and u_2 is connected and, as a consequence, $G[S]_w$ is connected. Therefore, S is a weakly connected dominating set and $\gamma_w(G) \leq |S| = |U| - 1$. Therefore, if $\gamma_w(G) = |U|$, then for every pair of vertices $u, v \in U$ such that $N(u) \cap N(v) \neq \emptyset$, it follows that $\{u, v\}$ is a cut set. □

Lemma 9. *Let $G = (U \cup W, E)$ be a bipartite graph where $|U| \leq |W|$. If $\gamma_w(G) < |U|$, then there exists a vertex belonging to W of degree greater than or equal to three.*

Proof. Let S be a $\gamma_w(G)$-set of G. From $\gamma_w(G) < |U|$ we deduce that $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$, due to the domination property of S. Let $S_u^{(0)} = S \cap U$ and $S_w^{(0)} = S \cap W$. By the connectivity of $G[S]_w$, there exist $u_1 \in U - S$, $u_1' \in S_u^{(0)}$ and $w_1 \in S_w^{(0)}$ such that $u_1 w_1 \in E$ and $u_1' w_1 \in E$. If $deg(w_1) \geq 3$, then we are done. If $deg(w_1) = 2$, then for $S_u^{(1)} = S_u^{(0)} \cup \{u_1\}$ and $S_w^{(1)} = S_w^{(0)} - \{w_1\}$ we have that $S^{(1)} = S_u^{(1)} \cup S_w^{(1)}$ is a $\gamma_w(G)$-set.

As above, there exist $u_2 \in U - S^{(1)}$, $u_2' \in S_u^{(1)}$ and $w_2 \in S_w^{(1)}$ such that $u_2 w_2 \in E$ and $u_2' w_2 \in E$. If $deg(w_2) \geq 3$, then we are done. If $deg(w_2) = 2$, then for $S_u^{(2)} = S_u^{(1)} \cup \{u_2\}$ and $S_w^{(2)} = S_w^{(1)} - \{w_2\}$ we have that $S^{(2)} = S_u^{(2)} \cup S_w^{(2)}$ is a $\gamma_w(G)$-set.

By repeating this argument we conclude that either there exists $w_k \in S \cap W$ such that $deg(w_k) \geq 3$ or there exists a $\gamma_w(G)$-set, say D, such that $D \cap W = \emptyset$, which is a contradiction. The proof is complete. □

A matching $M \subseteq E$ from U to W is a set of $|U|$ independent edges in G.

Theorem 10 (P. Hall, 1935)**.** *A bipartite graph $G = (U \cup W, E)$ contains a matching from U to W if and only if $|N(X)| \geq |X|$, for every $X \subseteq U$.*

We will use Hall's theorem to prove the following result.

Proposition 11. *For any semiregular bipartite graph $G(U \cup W, E)$, $\tau(G) = \min\{|U|, |W|\}$.*

Proof. In a (δ_1, δ_2)-semiregular bipartite graph $G = (U \cup W, E)$ every vertex of U has degree δ_1 and every vertex of W has degree δ_2. We suppose $|U| \leq |W|$ and, as a consequence, $\delta_1 \geq \delta_2$. For every $X \subset U$ we have $\delta_1|X| \leq \delta_2|N(X)| \leq \delta_1|N(X)|$. Hence, $|X| \leq |N(X)|$. Thus, by Hall's theorem we conclude that every maximum matching of G has size $|U|$. As a consequence, by König-Egerváry's theorem we conclude $\tau(G) = |U|$. □

Proposition 12. *Let $G = (U \cup W, E)$ be a bipartite semiregular graph where $|U| \leq |W|$. If $\gamma_w(G) = \tau(G)$, then for every pair of vertices $u, v \in U$ such that $N(u) \cap N(v) \neq \emptyset$, it follows that $\{u, v\}$ is a cut set.*

Proof. Assume that $G = (U \cup W, E)$ is a (δ_1, δ_2)-semiregular bipartite graph which satisfies the premises. For $\delta_2 = 2$ the result is straightforward, while for $\delta_2 \geq 3$ the result is a direct consequence of Lemma 8 and Proposition 11. $\square$

Lemma 13. *Let $G = (U \cup W, E)$ be a connected (δ_1, δ_2)-semiregular bipartite graph where $|U| \leq |W|$. If for every pair of vertices $u, v \in U$ such that $N(u) \cap N(v) \neq \emptyset$, it follows that $\{u, v\}$ is a cut set, then $\delta_2 \leq 2$.*

Proof. Among all the pairs of vertices $a, b \in U$ such that $N(a) \cap N(b) \neq \emptyset$, we take a pair, say u, v, that minimizes

$$\mu = \min_{i \in \{1,\ldots,k\}} \{|W_i|\},$$

where $G_1 = (U_1 \cup W_1, E_1), \ldots, G_k = (U_k \cup W_k, E_k)$ are the connected components of $G - \{u, v\}$. Let $w \in W$ be such that $u, v \in N(w)$. Let us assume, without loss of generality, that $\mu = |W_1|$.

Note that, each vertex in W_1 only has neighbors in U_1 or in $\{u, v\}$. By the connectivity of G we have $|N_{W_1}(u)| < \delta_1$ or $|N_{W_1}(v)| < \delta_1$.

We proceed by contradiction. Suppose $\delta_2 \geq 3$. Then, since $|W_1| \geq \delta_1$, there exists at least one vertex $w' \in W_1$ such that $|N_{U_1}(w')| \geq 2$ (otherwise, every vertex of W_1 must be adjacent to u and v and, as a consequence, $|N_{W_1}(u)| = |N_{W_1}(v)| = \delta_1$, which is a contradiction). Let $u', v' \in U_1$ be such that $u', v' \in N(w')$. If $G_1 - \{u', v'\}$ is a connected graph, then $G - \{u', v'\}$ also is a connected graph, a contradiction. So, $G_1 - \{u', v'\}$ is not connected. Let $G'_1 = (U'_1 \cup W'_1, E'_1), \ldots, G'_{k'} = (U'_{k'} \cup W'_{k'}, E'_{k'})$ be the connected components of $G_1 - \{u', v'\}$. Now, given $i, j \in \{1, ..., k'\}$, $i \neq j$, if there exist $x \in W'_i$ and $y \in W'_j$ such that $x \in N(u)$ and $y \in N(v)$, or vice versa, then x and y belong to the same component of $G - \{u', v'\}$, *i.e.*, x, u, w, v, y is a path in $G - \{u', v'\}$. Thus, as $G - \{u', v'\}$ is not connected, there exists $i \in \{1, ..., k'\}$, such that $x \notin N(u) \cup N(v)$, for every $x \in W'_i$. So, $G'_i = (U'_i \cup W'_i, E'_i)$ is a connected component of $G - \{u', v'\}$ and $|W'_i| < |W_1|$, which is a contradiction with the minimality of $|W_1|$. $\square$

Theorem 14. *Let $G = (U \cup W, E)$ be a connected (δ_1, δ_2)-semiregular bipartite graph. Then $\gamma_w(G) = \tau(G)$ if and only if $\min\{\delta_1, \delta_2\} \leq 2$.*

Proof. Let us suppose $|U| \leq |W|$ and, as a consequence, $\delta_1 \geq \delta_2$. If $\gamma_w(G) = \tau(G)$, then by Proposition 12 and Lemma 13 we deduce $\delta_2 \leq 2$.

Conversely, if $\delta_2 = 1$, then G is a star graph and $\gamma_w(G) = \tau(G) = 1$. Moreover, if $\delta_2 = 2$, then by Lemma 9 we obtain $\gamma_w(G) = |U|$. So, by Proposition 11 we conclude $\gamma_w(G) = \tau(G)$. $\square$

4. Regular graphs

Lemma 15. *Let G be a regular graph of order n. Then $\gamma_w(G) = \tau(G)$ if and only if G is bipartite and $\gamma_w(G) = \frac{n}{2}$.*

Proof. Let $G = (V, E)$ be a δ-regular graph of order n. We suppose $\gamma_w(G) = \tau(G)$. Let S be a $\tau(G)$-set. Note that S is a weakly connected dominating set and $\overline{S} = V - S$ is an independent set. Hence, the size of G is

$$\frac{n\delta}{2} = \sum_{v\in S} |N_{\overline{S}}(v)| + \frac{1}{2}\sum_{v\in S} |N_S(v)|.$$

Now, since $\sum_{v\in S} |N_{\overline{S}}(v)| = (n - |S|)\delta$, we obtain

$$|S|\delta = \frac{1}{2}\left(\delta n + \sum_{v\in S} |N_S(v)|\right). \tag{1}$$

Thus, if $\sum_{v\in S} |N_S(v)| > 0$, then $\gamma_w(G) = |S| > \frac{n}{2}$, a contradiction. Therefore, $\sum_{v\in S} |N_S(v)| = 0$ and, as a consequence, G is bipartite and by (1) we have $\gamma_w(G) = |S| = \frac{n}{2}$.

Conversely, suppose G is a bipartite graph where $\gamma_w(G) = \frac{n}{2}$. Since for bipartite graphs the size of a maximum matching equals the size of a minimum vertex cover (König-Egerváry's theorem), and every δ-regular bipartite graph ($\delta \geq 1$) has a perfect matching, we conclude $\tau(G) = \frac{n}{2}$. The proof is complete. $\square$

As a direct consequence of Lemma 15 and Theorem 14 we obtain the following result.

Theorem 16. *Let G be a regular graph of order n. Then $\gamma_w(G) = \tau(G)$ if and only if G is isomorphic to a cycle graph of even order.*

5. Unicyclic graphs

Let G be a connected unicyclic graph and let $C = \{u_1, u_2, ..., u_k\}$ be the set of vertices belonging to the cycle C_k of G. We suppose that u_i is adjacent to u_{i+1}, for every $i \in \{1, 2, ..., k\}$. Here the subscripts are taken modulo k. Let $G_i = G - \{u_{i-1}u_i, u_iu_{i+1}\}$ be the subgraph obtained from G by removing the edges $u_{i-1}u_i$ and u_iu_{i+1}. Let $T_i = (V_i, E_i)$ be the connected component of G_i containing u_i. Note that T_i is a tree which can be seen as a tree with root u_i. Let $\mathcal{F} = \{T_i : \ i \in \{1, ...k\}\}$. We say that T_i belongs to the family $\mathcal{F}_0 \subset \mathcal{F}$ if T_i is a trivial graph or if the root u_i does not belong to any $\gamma_w(T_i)$-set. Also, we define the family $\mathcal{F}_1$ as $\mathcal{F}_1 = \{T_i \in \mathcal{F} : \ T_i \notin \mathcal{F}_0\}$, *i.e.*, $T_i \in \mathcal{F}_1$ if and only if there exists a $\gamma_w(T_i)$-set, S_i, such that $u_i \in S_i$ and T_i is not a trivial tree. Note that there exists a $\gamma_w(G)$-set, S, such that $u_l \in S$, for every $T_l \in \mathcal{F}_1$.

With the above notation we establish the following result.

Theorem 17. *Let G be a connected unicyclic graph. The following assertions hold.*

(i) *If* $|\mathcal{F}_1| \geq 2$, *then* $\gamma_w(G) = \tau(G)$ *if and only if for every two trees* $T_i, T_j \in \mathcal{F}_1$ *such that* $T_{i+1}, T_{i+2}, ..., T_{j-1} \in \mathcal{F}_0$, *it follows* $j - i \equiv 0 \pmod 2$.

(ii) *If* $\mathcal{F}_1 = \emptyset$ *or* $|\mathcal{F}_1| = 1$, *then* $\gamma_w(G) = \tau(G)$ *if and only if the cycle of* G *has even order.*

Proof. To prove the necessity of (i) we proceed by contradiction. We suppose there exist two trees $T_i, T_j \in \mathcal{F}_1$ such that $T_{i+1}, T_{i+2}, ..., T_{j-1} \in \mathcal{F}_0$ and the path $P_k = (V_p, E_p)$ induced by the vertices $u_{i+1}, u_{i+2}, ..., u_{j-1}$ has even order k. Let S be a $\tau(G)$-set (which also is a $\gamma_w(G)$-set) such that $u_l \in S$, for every $T_l \in \mathcal{F}_1$. Now, let $S' = S - V_p$, $S'' = S' \cup \{u_{i+3}, u_{i+5}, ..., u_{j-2}\}$, for $j \geq i+5$, and $S'' = S'$, for $j = i+3$. Let us show that S'' is a weakly connected dominating set. If $v \in V_p - S''$, then there exists $u \in \{u_i, u_{i+3}, u_{i+5}, ..., u_{j-2}\} \subset S''$ such that u and v are adjacent. If $v \in V - (V_p \cup S)$, then there exists $v' \in S' \subset S''$, such that v' and v are adjacent. Hence, S'' is a dominating set. On the other hand, $u_{i+1}u_{i+2}$ is not an edge of $G[S'']_w$ but, as $G[S]_w = G$ because S is a cover set, there is a path in $G[S'']_w$ from u_{i+2} to u_{i+1}. Moreover, as $T_{i+1}, T_{i+2}, ..., T_{j-1} \in \mathcal{F}_0$, for every u_l such that $l \in \{i+1, i+2, ..., j-1\}$ and T_l is not trivial, there exists at least one vertex of T_l belonging to $S \cap S''$ which is a neighbor of u_l. Therefore, $G[S'']_w$ is connected. As a consequence, S'' is a weakly connected dominating set. We have $|S'' \cap V_p| = \frac{k}{2} - 1$ and $|S \cap V_p| = \tau(P_k) = \frac{k}{2}$, due to $G[S]_w = G$. Therefore, $|S''| < |S| = \gamma_w(G)$, which is a contradiction.

To prove the sufficiency of (i) we need to show that there exists a $\gamma_w(G)$-set of cardinality $\tau(G)$. Let D be a $\gamma_w(G)$-set such that $u_l \in D$, for every $T_l \in \mathcal{F}_1$. Let $T_i, T_j \in \mathcal{F}_1$ such that $T_{i+1}, T_{i+2}, ..., T_{j-1} \in \mathcal{F}_0$ and the path $P_k = (V_p, E_p)$ induced by the vertices $u_{i+1}, u_{i+2}, ..., u_{j-1}$ has odd order k. If $|D \cap V_p| \leq \frac{k-1}{2} - 1$, then there are at least two edges of P_k not covered by D, which is a contradiction with the connectivity of $G[D]_w$, so $|D \cap V_p| \geq \frac{k-1}{2}$. Since k is odd, there is a vertex cover of P_k of cardinality $\frac{k-1}{2}$ and, as a consequence, $|D \cap V_p| = \frac{k-1}{2} = \tau(P_k)$.

Now let $\{P_{k_1} = (V_{k_1}, E_{k_1}), P_{k_2} = (V_{k_2}, E_{k_2}), ..., P_{k_r} = (V_{k_r}, E_{k_r})\}$ be the set of paths obtained (by the procedure used above to define P_k) from the different pairs of trees $T_i, T_j \in \mathcal{F}_1$ such that $T_{i+1}, T_{i+2}, ..., T_{j-1} \in \mathcal{F}_0$. Let $\mathcal{F}_2$ be the set of non-trivial trees belonging to $\mathcal{F}_0$. Then we have

$$\begin{aligned}\gamma_w(G) = |D| &= \sum_{T_i \in \mathcal{F}_1} \gamma_w(T_i) + \sum_{T_i \in \mathcal{F}_2} \gamma_w(T_i) + \sum_{l=1}^{r} |D \cap V_{k_l}| \\ &= \sum_{T_i \in \mathcal{F}_1} \tau(T_i) + \sum_{T_i \in \mathcal{F}_2} \tau(T_i) + \sum_{j=1}^{r} \tau(P_{k_j}) \\ &\geq \tau(G).\end{aligned}$$

Therefore, $\gamma_w(G) = \tau(G)$. The proof of (i) is complete.

To prove (ii) we differentiate two cases.

Case 1: $\mathcal{F}_1 = \emptyset$. As above, $\mathcal{F}_2$ denotes the set of trees different from a trivial graph belonging to $\mathcal{F}_0$. In such a case, every $\gamma_w(G)$-set, S, satisfies

$$S = S_c \cup \left(\bigcup_{T_i \in \mathcal{F}_2} \gamma_w(T_i) \right),$$

where S_c is a $\gamma_w(C_k)$-set. Analogously, every $\tau(G)$-set, S', satisfies

$$S' = S'_c \cup \left(\bigcup_{T_i \in \mathcal{F}_2} \tau(T_i) \right),$$

where S'_c is a $\tau(C_k)$-set. Since for every tree T, it follows $\tau(T) = \gamma_w(T)$, we have $|S| = |S'|$ if and only if $\gamma_w(C_k) = \tau(C_k)$. Hence, by Claim 4 we conclude $\gamma_w(G) = \tau(G)$ if and only if k is even. Therefore, the proof of (ii) for this case is complete.
Case 2: $|\mathcal{F}_1| = 1$. If $T_i \in \mathcal{F}_1$, then $T_{i+1}, T_{i+2}, ..., T_{i-1} \in \mathcal{F}_0$. So, the proof of (ii) for this case is obtained by analogy to the proof of (i) by taking $T_j = T_i$. □

In order to complete the method that decides whether a given unicyclic graph satisfies $\gamma_w(G) = \tau(G)$, we provide a deterministic algorithm that determines whether $T_i \in \mathcal{F}_0$ or $T_i \in \mathcal{F}_1$. For the sake of generality, we recall both families of trees $\mathcal{F}_0$ and $\mathcal{F}_1$ in terms of a given root of the tree. This means that a tree T with root v, denoted by T_v, is in $\mathcal{F}_0$ if either T is a trivial graph or the root v does not belong to any $\gamma_w(T)$-set.

Proposition 18. *Let $T_v = (V, E)$ be a rooted tree with root $v \in V$. Let $T_1 = (V_1, E_1), \ldots, T_k = (V_k, E_k)$ be the rooted trees resulting from the k connected components of $T_v - \{v\}$, where the corresponding roots $v_1, \ldots, v_k$ are the vertices of T_v such that v_j is adjacent to v and $v_j \in V_j$, $j \in \{1, ..., k\}$. Let $F = \{T_j \in \mathcal{F}_0 : 1 \le j \le k\}$. Then, $T_v \in \mathcal{F}_0$ if and only if $|F| = 0$.*

Proof. Let S be a $\tau(T_v)$-set and let S_j be a $\tau(T_j)$-set. It should be noted that the set $(S - V_j) \cup S_j \cup \{v_j\}$ is a vertex cover of T. Therefore, if $v_j \in S_j$, then $|(S - V_j) \cup S_j \cup \{v_j\}| = |(S - V_j) \cup S_j| = \tau(T)$. On the other hand, if there does not exist a $\tau(T_j)$-set S_j such that $v_j \in S_j$, then either $v_j \in S$ and $|S \cap V_j| = \tau(T_j)+1$ or $v_j \notin S$ and $|S \cap V_j| = \tau(T_j)$. Consequently,

$$\begin{aligned}
\tau(T_v) =& |\{v\} \cap S| + \sum_{j=1}^{k} |S \cap V_j| \\
=& |\{v\} \cap S| + \sum_{j=1}^{k} \tau(T_j) + \sum_{T_j \in F} |\{v_j\} \cap S|.
\end{aligned}$$

Let us suppose $|F| = 0$. In this case for every $\tau(T_v)$-set S we have

$$\tau(T_v) = |\{v\} \cap S| + \sum_{j=1}^{k} \tau(T_j). \tag{2}$$

Note that for every $j \in \{1, ..., k\}$ we can take a $\tau(T_j)$-set S_j such that $v_j \in S_j$. In such a case, $\cup_{j=1}^k S_j$ is a $\tau(T_v)$-set, so $\tau(T_v) = \sum_{j=1}^k \tau(T_j)$. Thus, by Equation (2) we obtain that for every $\tau(T_v)$-set S, $v \notin S$. Therefore, $T_v \in \mathcal{F}_0$.

Conversely, if $T_v \in \mathcal{F}_0$, then for every $\tau(T_v)$-set S we have $v \notin S$ and, as a consequence, $v_j \in S$ for every $j \in \{1, ..., k\}$. Now, if there exists $T_j \in \mathcal{F}_0$, then for every $\tau(T_j)$-set S_j we have $v_j \notin S_j$ and, as a consequence, $|S \cap V_j| = |S_j| + 1$. Then by taking $S' = S - V_j$, we obtain that $S'' = S' \cup S_j \cup \{v\}$ is a $\tau(T_v)$-set which satisfies $v \in S''$, a contradiction. Therefore, $|F| = 0$. □

Proposition 18 states whether a rooted tree T_v is in $\mathcal{F}_0$. Therefore, Algorithm 1 recursively solves this decision problem in linear time complexity with respect to the number of vertices.

Algorithm 1 Deciding whether $T_v \in \mathcal{F}_0$

Require: A rooted tree $T_v = (V, E)$.
if T_v is a trivial graph **then**
  return true
end if
for all tree T_j child of v with root v_j **do**
  if $T_j \in \mathcal{F}_0$ **then**
    return false;
  end if
end for
return true

5.1. Spanning trees

The set of all spanning trees of a connected graph G is denoted by $\mathcal{S}_t(G)$.

Lemma 19. *Let G be a connected graph. For every $T \in \mathcal{S}_t(G)$, $\gamma_w(G) \leq \gamma_w(T)$ and $\tau(T) \leq \tau(G)$.*

Proof. The inequality $\gamma_w(G) \leq \gamma_w(T)$ immediately follows from the fact that every weakly connected dominating set of a spanning tree of G is a weakly connected dominating set of G. Analogously, the inequality $\tau(T) \leq \tau(G)$ follows from the fact that every vertex cover of G is a vertex cover of any spanning tree of G. □

Every spanning tree of a cycle graph C_k is a path graph P_k. For k even we have $\gamma_w(C_k) = \gamma_w(P_k) = \tau(P_k) = \tau(C_k) = \frac{k}{2}$ and, for k odd we have $\gamma_w(C_k) = \gamma_w(P_k) = \tau(P_k) = \frac{k-1}{2}$ while $\tau(C_k) = \frac{k+1}{2}$.

Proposition 20. *Let G be a connected unicyclic graph. If $\gamma_w(G) = \tau(G)$, then for every $T \in \mathcal{S}_t(G)$, $\tau(T) = \tau(G)$ and $\gamma_w(T) = \gamma_w(G)$.*

Proof. If $\gamma_w(G) = \tau(G)$, then Lemma 19 leads to $\tau(T) \leq \tau(G) = \gamma_w(G) \leq \gamma_w(T)$, for every $T \in \mathcal{S}_t(G)$. Now, since for every tree, $\gamma_w(T) = \tau(T)$, we conclude $\tau(T) = \tau(G) = \gamma_w(G) = \gamma_w(T)$. $\square$

Given two adjacent vertices x, y of G, we denote by $G - \{xy\}$ the subgraph obtained by removing from G the edge xy.

Lemma 21. *Let G be a connected unicyclic graph. If $\gamma_w(G) < \tau(G)$, then the following assertions hold.*

(i) *For every $\gamma_w(G)$-set S, $G[S]_w \in \mathcal{S}_t(G)$.*

(ii) *There exists $T \in \mathcal{S}_t(G)$, such that $\tau(G) = \tau(T) + 1$.*

Proof. By definition, for every weakly connected dominating set S of G, $G[S]_w$ is a spanning subgraph of G and which is connected. Moreover, since every vertex cover is a weakly connected dominating set, if $\gamma_w(G) < \tau(G)$, then for every $\gamma_w(G)$-set S, there exist two adjacent vertices x, y of G such that $x, y \notin S$. Let C be the cycle of G. By the connectivity of $G[S]_w$ we deduce that $x, y \in C$. So, $G[S]_w$ is a spanning tree of G, *i.e.*, $G[S]_w = G - \{xy\}$. Therefore, (i) follows.

Moreover, for $T = G - \{xy\} = G[S]_w$ we have $\gamma_w(G) = \gamma_w(T) = \tau(T)$. Thus, from $\tau(G) > \gamma_w(G)$ we deduce $\tau(G) > \tau(T)$. Note that, by the connectivity of $G[S]_w$, there is no edge $x'y'$ different from xy such that $x', y' \notin S$. Hence, $S \cup \{x\}$ is a vertex cover for G and we conclude $\tau(G) = \tau(T) + 1$. The proof is complete. $\square$

By Proposition 20 and Lemma 21 (ii) we obtain the following result.

Theorem 22. *Let G be a connected unicyclic graph. Then $\gamma_w(G) = \tau(G)$ if and only if $\tau(G) = \tau(T)$, for every $T \in \mathcal{S}_t(G)$.*

To apply Theorem 22 for unicyclic graphs where the cycle is even we can use the shortest augmenting path algorithm to compute maximum matching (see, for instance, [5]) and then, by König-Egerváry's theorem, we obtain the value of $\gamma_w(G)$.

6. Block graphs

A graph is a *block graph* if it is connected and every block (maximal 2-connected component) is a clique (a complete subgraph). Note that every block graph can be constructed from a tree by replacing every edge by a clique of arbitrary size; any two cliques have at most one vertex in common. So, every tree is a block graph.

We know that for every tree T it follows $\tau(T) = \gamma_w(T)$. So, from now on G denotes a block graph different from a tree. Let $C = (U, E)$ be a block of G where $U = \{u_1, u_2, ..., u_k\}$ and $k \geq 3$. Let $\mathcal{F} = \{G_1 = (U_1, E_1), \ldots, G_k = (U_k, E_k)\}$ be the set of connected components resulting from removing the edges of C. We assume that $u_i \in U_i$, for every $i \in \{1, ..., k\}$. With this notation we establish the following result.

Proposition 23. *If $\tau(G) = \gamma_w(G)$, then for every $i \in \{1, ..., k\}$ either $\tau(G_i) = \gamma_w(G_i)$ or G_i is a trivial graph.*

Proof. Let W be a $\tau(G)$-set and let $W_i = W \cap U_i$. It should be noted that W_i is a vertex cover of the corresponding non-trivial graph G_i. Let us assume, without loss of generality, that G_1 is a non-trivial graph with $\tau(G_1) > \gamma_w(G_1)$ and let S_1 be a $\gamma_w(G_1)$-set. Since $k \geq 3$ and C is a complete graph, all but one vertex of $U = \{u_1, u_2, ..., u_k\}$ belong to the vertex cover W. Then, $S = S_1 \cup W_2 \cup \cdots \cup W_k$ is a weakly connected dominating set of G. Since $|S_1| < |W_1|$ by assumption, and $|S| = |W| - |W_1| + |S_1|$, we have $|S| < |W|$, a contradiction. □

Now, we proceed similarly to the analysis for unicycle graphs by defining two families $\mathcal{F}_0 \subseteq \mathcal{F}$ and $\mathcal{F}_1 \subseteq \mathcal{F}$. We say that G_i is in $\mathcal{F}_0$ if it is a trivial graph or if $\tau(G_i) = \gamma_w(G_i)$ and u_i does not belong to any $\tau(G_i)$-set. On the other hand, G_i is in $\mathcal{F}_1$ if $\tau(G_i) = \gamma_w(G_i)$ and there exists a $\tau(G_i)$-set, S_i, such that $u_i \in S_i$. Note that, $\mathcal{F}_0$ and $\mathcal{F}_1$ do not necessarily form a partition of $\mathcal{F}$. With this notation we establish the following result.

Theorem 24. $\tau(G) = \gamma_w(G)$ *if and only if* $|\mathcal{F}_0| \leq 1$ *and for every non-trivial graph* G_i, $\tau(G_i) = \gamma_w(G_i)$.

Proof. First, let us assume that $\tau(G) = \gamma_w(G)$. In this case, by Proposition 23 we have that $\tau(G_i) = \gamma_w(G_i)$ for every non-trivial graph G_i.

Let W be a $\tau(G)$-set and let $W_i = W \cap U_i$. Notice that W_i is a weakly connected dominating set of the non-trivial graph G_i. Suppose, without loss of generality, that $G_1, G_2 \in \mathcal{F}_0$. Now, for $j \in \{1, 2\}$, let S_j be a $\tau(G_j)$-set if G_j is a non-trivial graph and let $S_j = \emptyset$ if G_j is a trivial graph. Since C is a complete graph we have that all but one vertex of $U = \{u_1, u_2, ..., u_k\}$ belong to the vertex cover W, so $S = S_1 \cup S_2 \cup W_3 \cup \cdots \cup W_k \cup \{u_k\}$ is a weakly connected dominating set of G. Note that we add u_k to S in case u_k is not in W. Thus, if $u_k \notin W$, then $u_1, u_2 \in W$ and, as a consequence, $|S_1| < |W_1|$ and $|S_2| < |W_2|$. Moreover, if $u_k \in W$, then $u_1 \in W$ or $u_2 \in W$ and, as a consequence, $|S_1| < |W_1|$ or $|S_2| < |W_2|$. In both cases we deduce $|S| < |W|$, a contradiction. Therefore, $|\mathcal{F}_0| \leq 1$.

Now, let us assume that $\tau(G_i) = \gamma_w(G_i)$ for every non-trivial graph G_i and at most one of them, say G_1, is in $\mathcal{F}_0$. This means that every subgraph G_i for $i \in \{2, ..., k\}$ is non-trivial. Let S be a $\gamma_w(G)$-set and let $S_i = S \cap U_i$, as before, every S_i is a weakly connected dominating set of the non-trivial graph G_i. Therefore,

$$|S| \geq |S_1| + \sum_{i=2}^{k} \gamma_w(G_i) = |S_1| + \sum_{i=2}^{k} \tau(G_i).$$

So we have

$$\sum_{i=2}^{k} \tau(G_i) \leq |S| - |S_1|.$$

On the other hand, for $i \in \{2, ..., k\}$ let W_i be a $\tau(G_i)$-set such that $u_i \in W_i$. Also, let $W_1 = \emptyset$ if G_1 is a trivial graph and let W_1 be a $\tau(G_1)$-set if G_1 is a non-trivial graph. Then, $W = W_1 \cup \cdots \cup W_k$ is a vertex cover of G and $|W| = |W_1| + \sum_{i=2}^{k} \tau(G_i) \leq |W_1| + |S| - |S_1|$. However, it should be noted that if G_1 is a trivial graph, then $|W_1| = 0 \leq |S_1|$, otherwise $|W_1| = \tau(G_1) = \gamma_w(G_1) \leq |S_1|$. So, $|W| \leq |S| = \gamma_w(G) \leq \tau(G)$ and, as a consequence, $|W| = |S| = \tau(G) = \gamma_w(G)$. □

Theorem 24 leads to Algorithm 2 that determines whether $\tau(G) = \gamma_w(G)$ for block graphs.

Algorithm 2 Determining whether $\tau(G) = \gamma_w(G)$ for a block graph G.

Require: A block graph G.

Let $C = (U, E)$ be a block of G of maximum order, where $U = \{u_1, u_2, ..., u_k\}$.
if $k = 2$ **then**
    return true. // Note that in this case G is a tree.
end if
Let $G_1 = (U_1, E_1), \ldots, G_k = (U_k, E_k)$ be the connected components resulting from removing the edges of C.
Call Algorithm 3 so as to build two sets $F_0 = \{G_i : G_i \in \mathcal{F}_0\}$ and $F_1 = \{G_i : G_i \in \mathcal{F}_1\}$.
if $|F_0| \leq 1$ and $|F_0| + |F_1| = k$ **then**
    return true.
else
    return false.
end if

7. Corona graphs

Let G and H be two graphs of order n_1 and n_2, respectively. Recall that the corona product $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G. We will denote by $V = \{v_1, v_2, ..., v_n\}$ the set of vertices of G and by $H_i = (V_i, E_i)$ the copy of H such that $v_i \sim v$ for every $v \in V_i$. We denote by N_k the null graph of order k.

Proposition 25. *For any connected graph G, $\gamma_w(G \circ H) = \tau(G \circ H)$ if and only if $H \cong N_k$.*

Proof. Let $G = (V, E)$. We know that V, is a $\gamma(G \circ H)$-set. Since $G \circ H[V]_w$ is connected, we have that $\gamma_w(G \circ H) = |V|$. Thus, if $H \cong N_k$, then V is a vertex cover of $G \circ H$. Hence, $\gamma_w(G \circ H) = \tau(G \circ H)$. Now, let W be a $\tau(G \circ H)$-set. If the size of H is different from zero, then for each vertex $v \in V$, the vertex cover W contains at least two vertices belonging to the copy of H corresponding to v. Hence, $\tau(G \circ H) > |V| = \gamma_w(G \circ H)$. The proof is complete. □

Algorithm 3 Determining if a block graph $G = (U, E)$ with extreme vertex u belongs to $\mathcal{F}_0$, to $\mathcal{F}_1$, or to none of them.

Require: G a block graph and u an extreme vertex in G.

```
if G is a trivial graph then
    return G belongs to F_0.
end if
Let C = ({u_1, ..., u_k}, E') be the block of G containing u.
Let G_1 = (U_1, E_1), ..., G_k = (U_k, E_k) be the connected components of G resulting from
    removing the edges of C from G, such that u_i ∈ U_i ∀i ∈ {1, ..., k}.
Let F_0 and F_1 be two sets.
for i ∈ {1, ..., k} do
    Recursively call Algorithm 3 on input the block graph G_i and the extreme vertex u_i.
    if G_i ∈ F_0 then
        F_0 = F_0 ∪ {G_i}
    else if G_i ∈ F_1 then
        F_1 = F_1 ∪ {G_i}
    end if
end for
if k = 2 and |F_0| = 2 then
    return G belongs to F_1. //interchangeably either u_1 or u_2 needs to be in a τ(G)-set.
end if
if k = 2 and |F_0| = |F_1| = 1 then
    return G_i belongs to F_0. //none τ(G)-set contains u.
end if
if |F_0| > 1 or |F_0| + |F_1| < k then
    return G_i does not belong neither to F_0 nor to F_1. //applying Theorem 24
else
    return G_i belongs to F_0. //Applying Theorem 24 and considering that the isolated
        vertex u is that graph belonging to F_0.
end if
```

Bibliography

[1] Y.Z. Chen, A.L. Liestman, Approximating minimum size weakly connected dominating sets for clustering mobile ad hoc networks, *Proceedings of the Third ACM International Symposium on MobileAd Hoc Networking and Computing* (MobiHoc'2002), June 2002, pp.157–164.

[2] J. Dunbar, J. Grossman, J. Hattingh, S. T. Hedetniemi, A. McRae, On weakly connected domination in graphs, *Discrete Mathematics* 167–168 (1997).

[3] R.G. Downey and M.R. Fellows, Fixed parameter tractability and completeness II: completeness for W[1], *Theory, Comp. Sci.*, vol. 141, (1995), pp.1-2.

[4] F. Harary, J. P. Hayes and H. Wu, A survey of the theory of hypercube graphs. *Computers & Mathematics with Applications* **15** (4) (1988) 277–289.

[5] A. Levitin, *Introduction to the Design and Analysis of Algorithms.* Addison Wesley, 2006.

[6] S. Pirzada, A. Dharwadker, Applications of Graph Theory, *Journal of the Korean Society for industrial and applied mathematics*, vol. 11, no. 4 (2007).

[7] R. Rajaraman, Topology control and routing in ad hoc networks: a survey, *SIGACT News*, Vol. 33, Number 2, pages 60–73, 2002.

[8] J. Wu, H. Li, On calculating connected dominating set for efficient routing in ad hoc wireless networks, *Proceedings of the 3rd International Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications*, ACM, (1999) 7–14.

[9] J. Wu, Extended Dominating-Set-Based Routing in Ad Hoc Wireless Networks with Unidirectional Links,*IEEE Trans. Parallel and Distributed Systems*, vol. 9, no. 3, (2002) 189–200.

[10] J. Wu, F. Dai, An Extended Localized Algorithm for Connected Dominating Set Formation in Ad Hoc Wireless Networks, *IEEE Transactions on Parallel and Distributed Systems*, vol. 15, no. 10, (2004).

[11] X. Xu and J. Ma, An efficient simulated annealing algorithm for the minimum vertex cover problem, *Neurocomputing*, vol.69, (2006), pp. 913–916.