# Game Theory Seminar

27.05.08

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UNIVERSITÉ DU LUXEMBOURG

# Menu of today

- *Auctions (follow-up on Bayesian games)*
- *Mixed and Behavioral strategies in Extensive-form Games*

# Auctions

# Independent private values auctions (IPVA)

- n bidders
- one single indivisible object or good
- each player simultaneously submits a bid (nonnegative real number)
- the highest bidder wins
- losers do not pay anything

# Epistemic aspects in IPVAs

- each player knows (privately) how much the object is worth to him
- each player considers the values of the object to the other players to be independent random variables from the interval [0,M]
- the probability distribution of these random variables is described by a given cumulative distribution F (increasing and differentiable)

# Cumulative distribution F

- F(v) is the probability that any of the players has a value for the object that is less than v
- E.g.:

$$F(v_i)^{n-1}$$

gives the probability for player i that all other players value the object less than he does

- F encodes the information about player types!

# IPVA as Bayesian games

- strategy profile: $b = (b_1 \ldots b_n)$
- type profile: $v = (v_1 \ldots v_n)$
- expected payoff:

$$u_i(b, v) = \begin{cases} v_i - b_i & \text{if } \{i\} = \text{argmax}_{j \in \{1,\ldots,n\}} b_j \\ 0 & \text{if } i \neq \text{argmax}_{j \in \{1,\ldots,n\}} b_j \end{cases}$$

# Finding Bayesian Equilibria in IPVAs (I)

- "we now show how to find a Bayesian equilibrium in which every player chooses his bid according to some function $\beta$ that is differentiable and increasing"

# Finding Bayesian Equilibria in IPVAs (II)

- Player $i$ expects other players' bids to be in $(0, \beta(M))$
- hence $b_i \leq \beta(M)$
- Suppose that his value is $v_i$ and he bids $\beta(w_i)$
- Another player $j$ submits a bid $b_j < \beta(w_i)$ iff $v_j$ is such that $\beta(v_j) < \beta(w_i)$
- hence iff $v_j < w_i$ since $\beta$ is increasing
- Therefore, the probability that $\beta(w_i)$ wins is $F(w_i)^{n-1}$
- and the expected payoff of $i$ from bidding $\beta(w_i)$ with value $v_i$ is:

$$(v_i - \beta(w_i))F(w_i)^{n-1}$$

# Finding Bayesian Equilibria in IPVAs (III)

- However, by the definition of an equilibrium, the optimal bid for $i$ with value $v_i$ should be $\beta(v_i)$

- hence, the derivative of the expected payoff w.r.t. $w_i$ should equal 0 when $w_i$ equals $v_i$:

$$0 = (v_i - \beta(v_i))F'(v_i)(n-1)F(v_i)^{n-2} - \beta'(v_i)F(v_i)^{n-1}$$

- This equation implies that, for any $x \in [0, M]$:

$$\beta(x)F(x)^{n-1} = \int_0^x y(n-1)F(y)^{n-2}F'(y)dy$$

- If types are uniformly distributed, i.e., for any $y \in [0, M], F(y) = y/M$, the formula above implies that, $\forall v_i \in [0, M]$:

$$\beta(v_i) = (1 - 1/n)v_i$$

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# Common value auctions (CVA)

- n bidders
- one single indivisible object
- each player simultaneously submits a bid (nonnegative real number)
- the highest bidder wins
- losers do not pay anything

# Epistemic aspects in CVAs

- the value of the good is the same for all bidders although they have different *estimations* of it (unknown common value)
- example ...

# Example of CVA (I)

- Two players: $1, 2$
- The value of the good depends on three independent random variables $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2$ taken from a uniform distribution on the interval $[0, 1]$
- The good is worth $A_0\tilde{x}_0 + A_1\tilde{x}_1 + A_2\tilde{x}_2$ where $A_0, A_1, A_2$ are given (commonly known by the bidders) nonnegative constants
- At the time of the auction, player 1 has observed $\tilde{x}_0, \tilde{x}_1$ and ignores $\tilde{x}_2$, while player 2 has observed $\tilde{x}_0, \tilde{x}_2$ and ignores $\tilde{x}_1$
- The two player types are, for player 1 $(\tilde{x}_0, \tilde{x}_1)$ and for player 2 $(\tilde{x}_0, \tilde{x}_2)$

# Example of CVA (II)

- Bids are denoted by $c_1$, respectively, $c_2$
- In case of tie, each player has 0.5 probability of getting the good at the price of his bid
- The utility payoff function is:

$$u_i(c_1, c_2, (\tilde{x}_0, \tilde{x}_1), (\tilde{x}_0, \tilde{x}_2)) = \begin{cases} A_0\tilde{x}_0 + A_1\tilde{x}_1 + A_2\tilde{x}_2 - c_i & \text{if } c_i > c_j, \\ (A_0\tilde{x}_0 + A_1\tilde{x}_1 + A_2\tilde{x}_2)/2 & \text{if } c_i = c_j, \\ 0 & \text{if } c_i < c_j. \end{cases}$$

- The only (linear) Bayesian equilibrium is given by the two bids:

$$A_0\tilde{x}_0 + 0.5(A_1 + A_2)\tilde{x}_1, A_0\tilde{x}_0 + 0.5(A_1 + A_2)\tilde{x}_2$$

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**it can be proven!**

# Example of CVA (III)

- Suppose that 1 expects 2 to bid according to the equilibrium, but he considers a different bid $b$ given the values $\tilde{x}_0 = x_0$ and $\tilde{x}_1 = x_1$ he has observed
- Bid $b$ would win the object for 1 if:

$$b > A_0 x_0 + 0.5(A_1 + A_2)\tilde{x}_2$$

i.e.,

$$2(b - A_0\tilde{x}_0)/(A_1 + A_2) > \tilde{x}_2$$

- Player 1 wins with $b$ with probability $Y(b) = 2(b - A_0 x_0)/(A_1 + A_2)$
- Notice that $Y(b) \in [0, 1]$

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# Example of CVA (IV)

- Hence, the conditionally expected payoff for 1 on the assumption that 2 plays his equilibrium strategy is:

$$\int_0^{Y(b)} (A_0x_0 + A_1x_1 + A_2y_2 - b)dy_2 = Y(b)(A_0x_0 + A_1x_1 + A_2Y(b)/2 - b)$$

- Notice that $Y(b)(A_0x_0 + A_1x_1 + A_2Y(b)/2 - b)$ is the conditionally expected value of the good, given that player 1's type is $(x_0, x_1)$ and that 1 could win by bidding $b$

- Substituting $Y(b)$ we get:

$$A_0x_0 + 0.5(A_1 + A_2)x_1$$

- Notice that substituting this value in the definition of $Y(b)$ we get $Y(b) = x_1$

- A similar argument can be provided for player's 2 optimal bid

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# ... instantiating the example (I)

- Suppose $A_0 = A_1 = A_2 = 100$ and $\tilde{x}_0 = 0, \tilde{x}_1 = 0.01$
- Recall that $\tilde{x}_2 \in [0,1]$ is unknown to player 1. Hence its expected value is 0.5
- Player 1's optimal bid is:

$$\begin{aligned} & A_0\tilde{x}_0 + 0.5(A_1 + A_2)\tilde{x}_1 \\ 0 + 0.5 \times 200 \times 0.01 &= 1 \end{aligned}$$

Notice that the expected value of the object is:

$$\begin{aligned} & A_0\tilde{x}_0 + A_1\tilde{x}_1 + A_2\tilde{x}_2 \\ 100 \times 0 + 100 \times 0.01 + 100 \times 0.5 &= 51 \end{aligned}$$

- The bid is less than 2% of the expected value!

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- The bid is less than 2% of the expected value!

**how is this possible?**

# ... instantiating the example (II)

- Recall first that $Y(b) = 2(b - A_0 x_0)/(A_1 + A_2)$
- Although the estimated value of the object is 51 the expected utility payoff of a bid $b = 50$ is

$$
\begin{aligned}
Y(b)(A_0\tilde{x}_0 + A_1\tilde{x}_1 + A_2 Y(b)/2 - b) \\
0.5(0 + 1 + 25 - 50) &= -12
\end{aligned}
$$

- Intuitively, "a bid of 50 would give player 1 a probability 0.5 of buying an object for 50 that would have an expected value of 26 when he gets to buy it at this price, so that 1's expected profit is indeed $0.5(26-50) = -12 < 0$"

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# The moral of the story

“When computing the expected profit from a particular bid in an auction, it is important that the bidder estimates the value of the object by its conditionally expected value given his current information *and the additional information that could be inferred if this bid won the auction*. This conditionally expected value is often significantly less than the expected value of the object given the bidder’s information at the time that he submits the bid. This fact is called the *winner’s curse*”

# ... instantiating the example (IV)

- Suppose now $A_0 = A_1 = \varepsilon$ and $A_2 = 100 - \varepsilon$
- In this case the equilibrium is:

$$A_0\tilde{x}_0 + 0.5(A_1 + A_2)\tilde{x}_1, A_0\tilde{x}_0 + 0.5(A_1 + A_2)\tilde{x}_2$$
$$\varepsilon\tilde{x}_0 + 50\tilde{x}_1, \varepsilon\tilde{x}_0 + 50\tilde{x}_2$$

- Although $\tilde{x}_0$ and $\tilde{x}_1$ have both small effects on the value of the object, the fact that only 1 knows $\tilde{x}_1$ has a big effect on 1's optimal bid
- As $\varepsilon$ goes to 0 the auction converges to a game where player 2 knows the real value of the good while player 1 only knows that the value was drawn from a uniform distribution over $[0, 100]$

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**exercises?**

# Sequential Equilibria of Extensive-form Games

# CHAPTER 3

# Mixed Strategies and Behavioral Strategies

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# Extensive-form games (recap)

A game in extensive form is a structure:

$$\Gamma^e = (\texttt{Tree}, \Omega, r, C, \texttt{chance}, N, \{S_i\}_{i \in N}, \{\texttt{Turn}_i\}_{i \in N}, M, \texttt{moves}, \{w_i\}_{i \in N})$$

where:

- $\texttt{Tree}$ is a tree on $S$ where the root is $r$ and the set of terminal nodes is $\Omega$
- $C$ is the set of chance nodes
- $\texttt{chance} : C \longrightarrow \Delta(\texttt{Tree}(C))$
- $N$ is the set of players
- $\{S_i\}_{i \in N}$ is a family of sets denoting the possible information states of each agent such that:

$$\forall i \in N : S_i \subset \mathcal{P}(S), \quad \bigcap_{i \in N} S_i = \emptyset, \quad \bigcup_{i \in N} S_i = S^*$$

- $\{\texttt{Turn}_i\}_{i \in N}$ is a family of sets denoting the states owned by each agent
- $\texttt{moves} : M \longrightarrow 2^{\texttt{Tree}}$ from the set of move labels $M$ to sets of edges in $\texttt{Tree}$
- $\{w_i\}_{i \in N}$ is the family of payoff functions: $w_i : \Omega \longrightarrow \mathbb{R}$

# ... more notation

Some more notation:

- The set of nodes belonging to player $i$ with information state $s$:

$$Y_s = s \cap \texttt{Turn}_i$$

- The set of all move labels of alternative branches following $Y_s$:

$$D_s = \{m \in M \mid \pi_1(\texttt{moves}(m)) \in Y_s\}$$

- The set of pure strategies of player $i$:

$$C_i = \times_{s \in S_i} D_s$$

# Strategies in Extensive-form games

To define strategies in extensive-form games we resort to their strategic-form representations:

1. *normal representation*
2. *multiagent representation*

# Normal representation (recap)

The normal representation $NR(\Gamma^e) = (N, \{C_i\}_{i \in N}, \{u_i\}_{i \in N})$ of $\Gamma^e$ is defined as follows:

- $N$ and $\{C_i\}_{i \in N}$ are the same
- $\{u_i\}_{i \in N}$ is defined from $\{w_i\}_{i \in N}$ as follows:

$$u_i(c) = \sum_{x \in \Omega} P(x|c) w_i(x)$$

where $P(\_|c)$ is inductively defined as follows:

**B:** if $x$ is the root, then $P(x|c) = 1$

**S:** if $(x, y) \in \texttt{Tree}$ and $y$ is a chance node with probability $q$, then $P(x|c) = qP(y|c)$
if $(x, y) \in \texttt{Tree}$ and $y$ is a choice node for $i$ in information state $r$ then:

$$P(x|c) = \begin{cases} P(y|c) & \text{if } c_i(r) \in D_r \\ 0 & \text{otherwise} \end{cases}$$

# Multiagent representation (recap)

The multiagent representation $MR(\Gamma^e) = (N, \{C_i\}_{i \in N}, \{v_i\}_{i \in N})$ of $\Gamma^e$ is defined as follows:

- $N = S^*$
- $\{C_i\}_{i \in N} = \{D_r\}_{r \in S^*}$
- $\{v_i\}_{i \in N}$ is defined from the set $\{u_i\}_{i \in N}$ in the normal representation. Functions $v_r : \times_{s \in S^*} D_s \longrightarrow \mathbb{R}$ are defined as follows:

  $\forall (d_s)_{s \in S^*} \in \times_{s \in S^*} D_s$: if $(c_j)_{j \in N}$ is the strategy profile for $NR(\Gamma^e)$ such that $\forall j \in N, t \in Sj : c_j(t) = d_t$, then:

$$v_r((d_s)_{s \in S^*}) = u_i((c_j)_{j \in N})$$

# Strategy profiles in extensive form

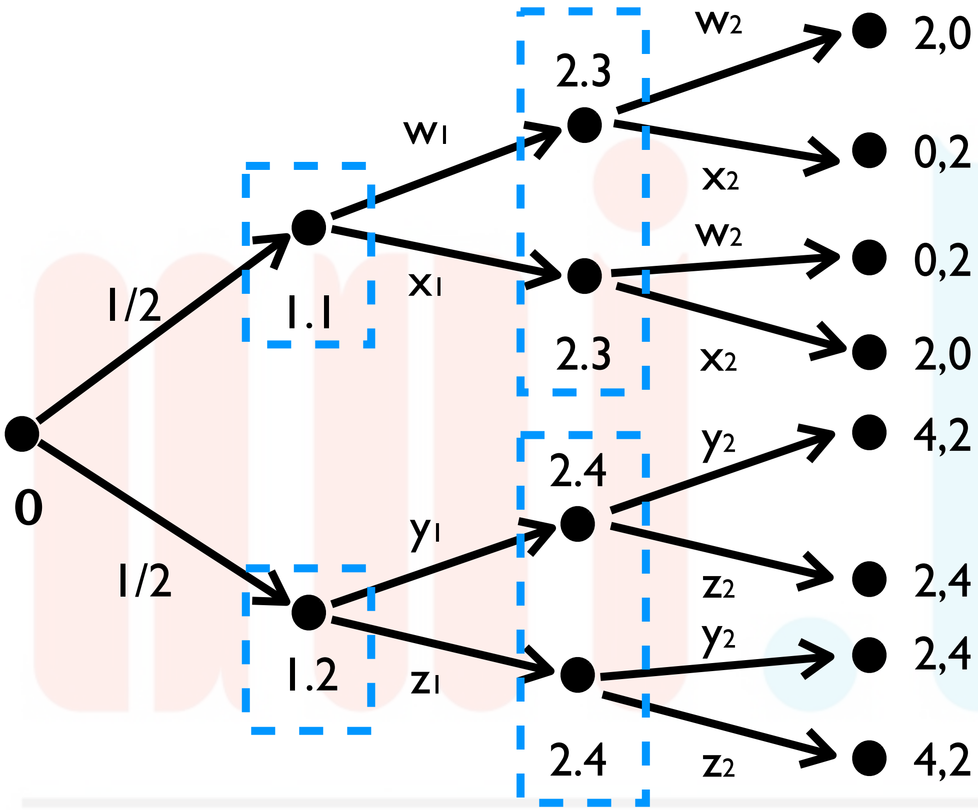
- A ***mixed-strategy profile*** is any randomized-strategy profile for the normal representation

$$\times_{i \in N} \Delta(C_i)$$

- A ***behavioral-strategy profile*** is any randomized-strategy for the multiagent representation

$$\times_{s \in S^*} \Delta(D_s) = \times_{i \in N} \times_{s \in S_i} \Delta(D_s)$$

# Mixed vs. behavioral




| $C_1$ \ $C_2$ | $w_2y_2$ | $w_2z_2$ | $x_2y_2$ | $x_2z_2$ |
|---|---|---|---|---|
| $w_1y_1$ | 3,1 | 2,2 | 2,2 | 1,3 |
| $w_1z_1$ | 2,2 | 3,1 | 1,3 | 2,2 |
| $x_1y_1$ | 2,2 | 1,3 | 3,1 | 2,2 |
| $x_1z_1$ | 1,3 | 2,2 | 2,2 | 3,1 |

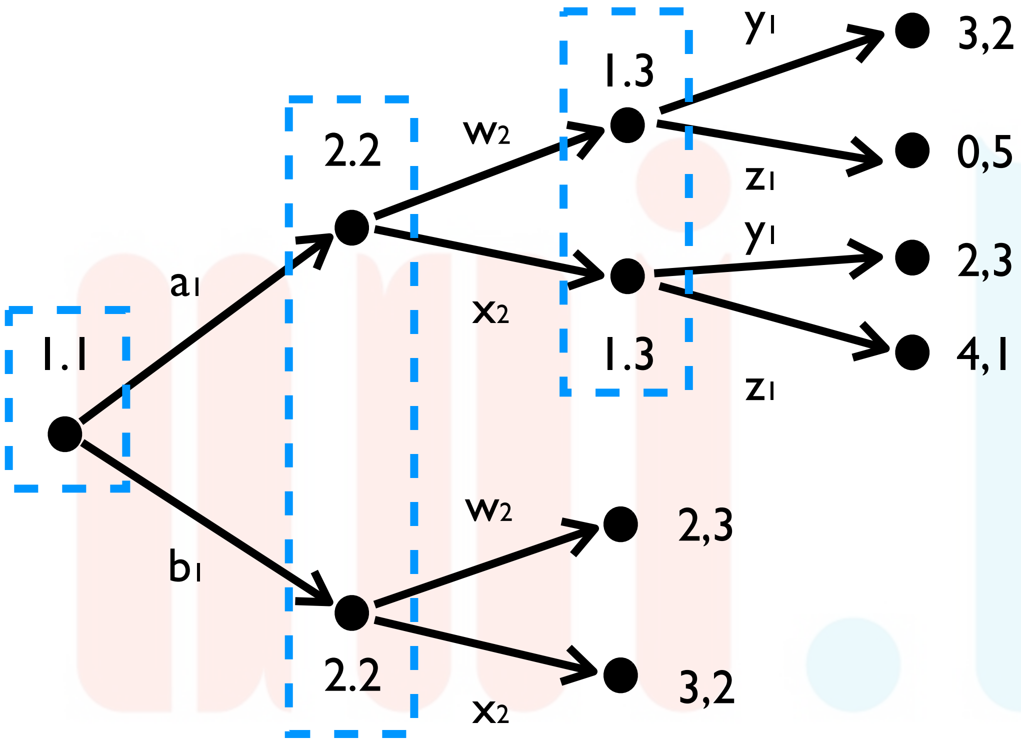
$\forall \alpha, \beta \in [0, 0.5]$ these profiles are equilibria of the normal representation:

$$(\alpha[w_1y_1]+\alpha[x_1z_1]+(0.5-\alpha)[w_1z_1]+(0.5-\alpha)[x_1y_1], \beta[w_2y_2]+\beta[x_2z_2]+(0.5-\beta)[w_2z_2]+(0.5-\beta)[x_2y_2])$$

All these equilibria are equivalent to the behavioral-strategy profile:

$$(0.5[w_1] + 0.5[x_1], 0.5[y_1] + 0.5[z_1], 0.5[w_2] + 0.5[x_2], 0.5[y_2] + 0.5[z_2])$$

# Mixed vs. behavioral




Notice that:

- The mixed strategy $0.5[a_1 y_1] + 0.5[b_1 z_1]$ for player 1 does not correspond to the behavioral strategy $(0.5[a_1] + 0.5[b_1], 0.5[y_1] + 0.5[z_1])$!
- Strategy $(0.5[a_1] + 0.5[b_1], [y_1])$ corresponds instead!
- Strategy $a_1 y_1$ is the only strategy compatible with information state 1.3

# Compatibility of inf. states and strategies

**Pure.** $\forall i \in N, c_i \in C_i, s \in S_i$: $s$ and $c_i$ are compatible iff $\exists c_{-i} \in C_{-i}$ such that:

$$\Sigma_{x \in Y_s} P(x|c) > 0$$

where $c = (c_{-i}, c_i)$

**Randomized.** $\forall i \in N, \tau_i \in \Delta(C_i), s \in S_i$: $s$ and $_i$ are compatible iff $\exists c_i \in C_i^*(s)$ such that:

$$\tau_i(c_i) > 0$$

where $C_i^*(s) = \{c_i \in C_i \mid c_i \text{ is compatible with } s\}$.

On $C_i^*(s)$ we can build the set $C_i^{**}(d_s, s) = \{c_i \in C_i^*(s) \mid c_i(s) = d_s\}$

# Representation of strategies: behavioral

A behavioral strategy $\sigma_i = (\sigma_{i.s})_{s \in S_i}$ for $i$ is a *behavioral representation* of a mixed strategy $\tau_i \in \Delta(C_i)$ iff $\forall s \in S_i, d_s \in D_s$:

$$\sigma_{i.s}(d_s) \left( \sum_{e_i \in C_i^*(s)} \tau_i(e_i) \right) = \sum_{c_i \in C^{**}_{(d_s,s)}(s)} \tau_i(c_i)$$

Intuitively, $\sigma_i$ is a behavioral representation of $\tau_i$ iff, for every move $d_s$ and every information state $s$ of $i$ which is compatible $\tau_i$, $\sigma_{i.s}(d_s)$ is the conditional probability that $i$ would choose $d_s$ at $s$ given that he chose a pure strategy that is compatible with $s$.

Any $\tau_i \in \Delta(C_i)$ has at least one behavioral representation in $\times_{s \in S_i} \Delta(D_s)$, and it might have more than one.

# Representation of strategies: mixed

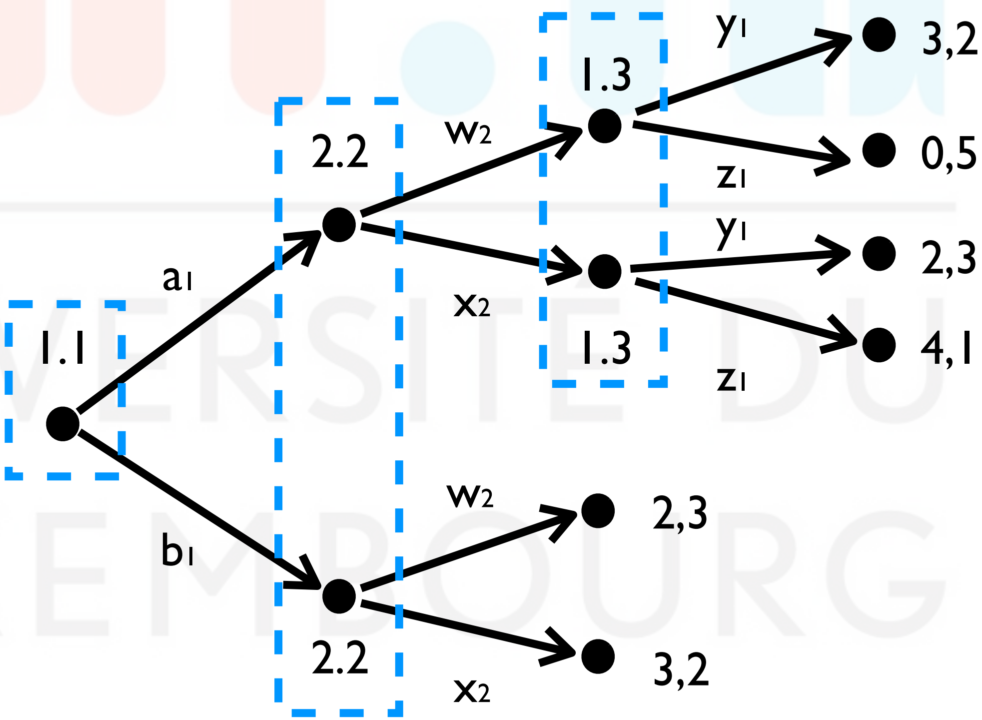
A mixed strategy $\tau_i \in \Delta C_i$ for $i$ is a *mixed representation* of a behavioral strategy $\sigma_i = (\sigma_{i.s})_{s \in S_i}$ iff $\forall c_i \in C_i$

$$\tau_i(c_i) = \prod_{s \in S_i} \sigma_{i.s}(c_i(s))$$

Intuitively, the mixed representation of a behavioral strategy $\sigma_i$ is the mixed strategy in $\Delta(C_i)$ in which $i$'s move at each information state $s$ has the marginal probability distribution $\sigma_{i.s}$ and is determined independently of his moves at all other information states.

# Equivalence between strategies (I)

- Two mixed strategies in $\Delta(C_i)$ are *behaviorally equivalent* iff they share a common behavioral representation
- E.g. $0.5[a_1 y_1] + 0.5[b_1 z_1]$ and $0.5[a_1 y_1] + 0.5[b_1 y_1]$ are behaviorally equivalent. Strategy $(0.5[w_1] + 0.5[x_1], [y_1])$ is the common behavioral representation



# Equivalence between strategies (II)

- Two mixed strategies in $\Delta(C_i)$ are *behaviorally equivalent* iff they share a common behavioral representation

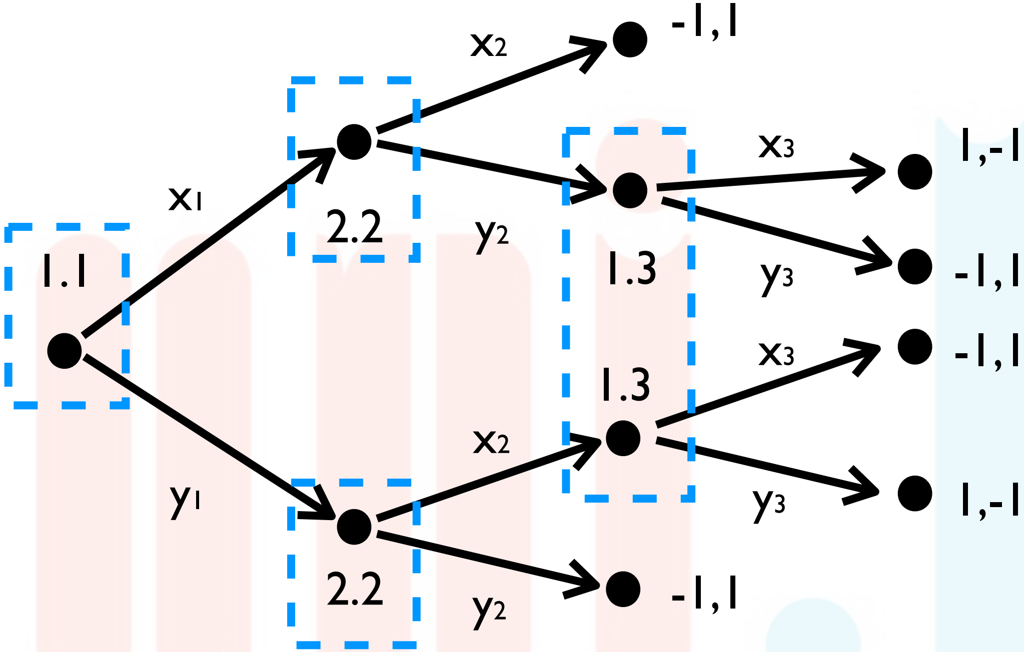
- E.g. $0.5[a_1y_1]+0.5[b_1z_1]$ and $0.5[a_1y_1]+0.5[b_1y_1]$ are behaviorally equivalent. Strategy $(0.5[w_1]+0.5[x_1],[y_1])$ is the common behavioral representation (and $0.5[a_1y_1]+0.5[b_1y_1]$ is its mixed representation)

- Two mixed strategies $\tau_1, \rho_i \in \Delta(C_i)$ are *payoff equivalent* iff, $\forall i \in N$ and $\tau_{-i} \in \times_{l \in N-i}$:

$$u_j(\tau_{-i}, \tau_i) = u_j(\tau_{-i}, \rho_i)$$

  where $u_j(\_)$ is $j's$ utility function in the normal representation of $\Gamma^e$. Intuitively, $\tau_1, \rho_i \in \Delta(C_i)$ are payoff equivalent if no player's expected utility depends on which of these two randomized strategies is used by player $i$.

# Equivalences related




**Theorem.**(Kuhn, 1953) If $\Gamma^e$ is a game with perfect recall, then any two mixed strategies in $\Delta(C_i)$ that are behaviorally equivalent are also payoff equivalent.

Why do we need perfect recall? Player 1's strategies $0.5[x_1x_3] + 0.5[y_1y_3]$ and $0.5[x_1y_3] + 0.5[y_1x_3]$ are behaviorally equivalent since they have the same behavioral representation $(0.5[x_1]+0.5[y_1], 0.5[x_3]+0.5[y_3])$. However, they are not payoff equivalent. Assuming 2 plays $[x_2]$ the first one gives payoff 0 and the second one $-1$.