Game Theory Seminar

10.06.08

davide.grossi@uni.lu Individual and Collective Reasoning Group

UNIVERSITÉ DU LUXEMBOURG

Menu of today

- *Backwards induction*
- *Extensions of the extensive (?!) form*
- *Iterated elimination of weakly dominated strategies*
- *Forward Induction*

Backwards induction

Recap

Subgame Perfect Equilibrium

A subgame perfect equilibrium is a profile s^* s.t. $\forall i \in N$ and $h \in H \backslash Z$ for which $P(h) = i$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h O_h(s^*_{-i}|_h, s^*_i|_h)$$

$\forall s_i$ for subgame $\Gamma(h)$.

The One Deviation Property

Let $\Gamma = \langle N, H, P, (\precsim_i) \rangle$ be a finite horizon extensive game with perfect information. The strategy profile s^* is a subgame perfect equilibrium of Γ iff $\forall i \in N$ and $h \in H \backslash Z$ for which $P(h) = i$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h O_h(s^*_{-i}|_h, s^*_i|_h)$$

for each s_i for subgame $\Gamma(h)$ that differs from $s^*_i|_h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

Kuhn's Theorem

Every finite extensive game with perfect information has a subgame perfect equilibrium

What's that for?

- Find a subgame perfect equilibrium in a game in extensive form
- Such equilibrium always exists if the game is finite (Kuhn's theorem)
- **NB**: that doesn't hold if we only require the game to have a finite horizon

How does it work?

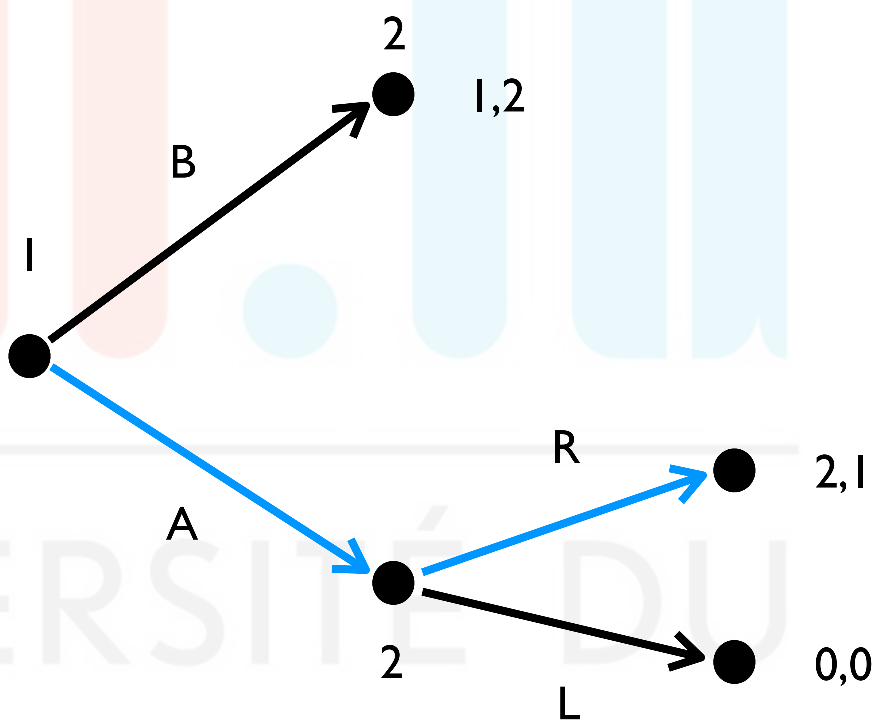
Let $\Gamma = \langle N, H, P, (\precsim_i) \rangle$ be a finite extensive game with perfect information. The subgame perferct equilibrium of Γ is built by induction on the length $l(\Gamma(h))$ of $\Gamma(h)$. For each length we define a function R associating a terminal history to every $h \in H$ and we show that such history is the outcome of the subgame perfect equilibrium of $\Gamma(h)$.

B: If $l(\Gamma(h)) = 0$, let $R(h) = h$

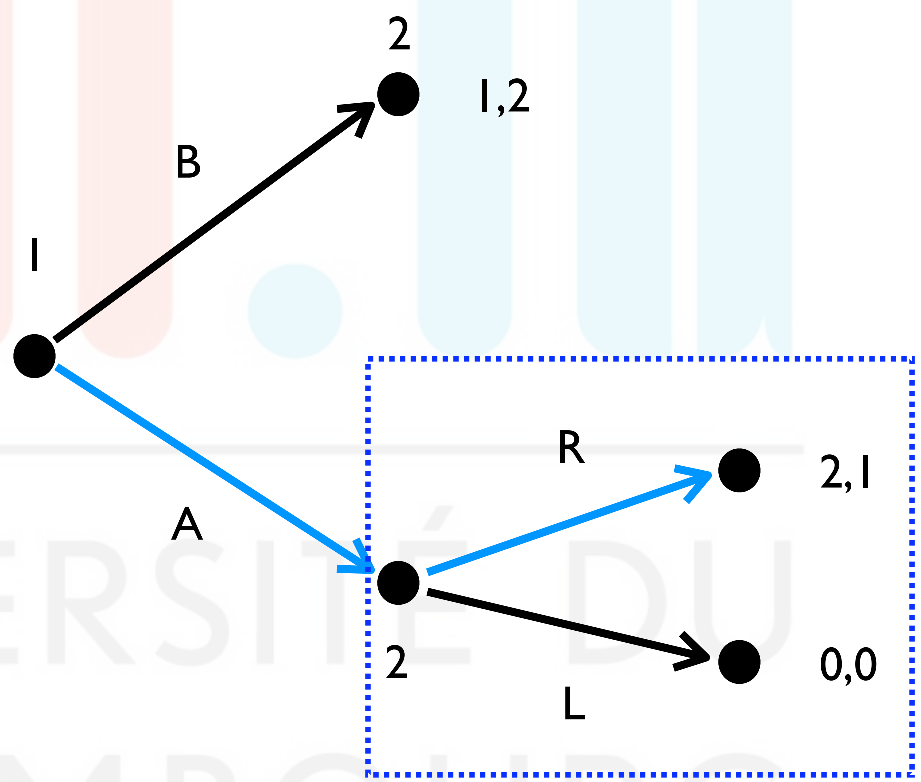
S: Suppose $R(h)$ is defined for all $h \in H$ such that $l(\Gamma(h)) \leq k$ with $0 \leq k$. Consider the history h^* such that $l(\Gamma(h^*)) = k + 1$ and $P(h^*) = i$. Notice that for all $a \in A(h^*)$ $l(\Gamma(h^*, a)) \leq k$. Now $s_i(h^*)$ is defined to $\precsim_i$-maximize $R(h^*, a)$ over $a \in A(h^*)$ and $R(h^*) = R(h^*, s_i(h^*))$.

This constructs a strategy profile for Γ. Since it $\precsim_i$-maximizes the outcome over the set of actions immediately available after a given history, for the One Deviation Property, it is a subgame perfect equilibrium.

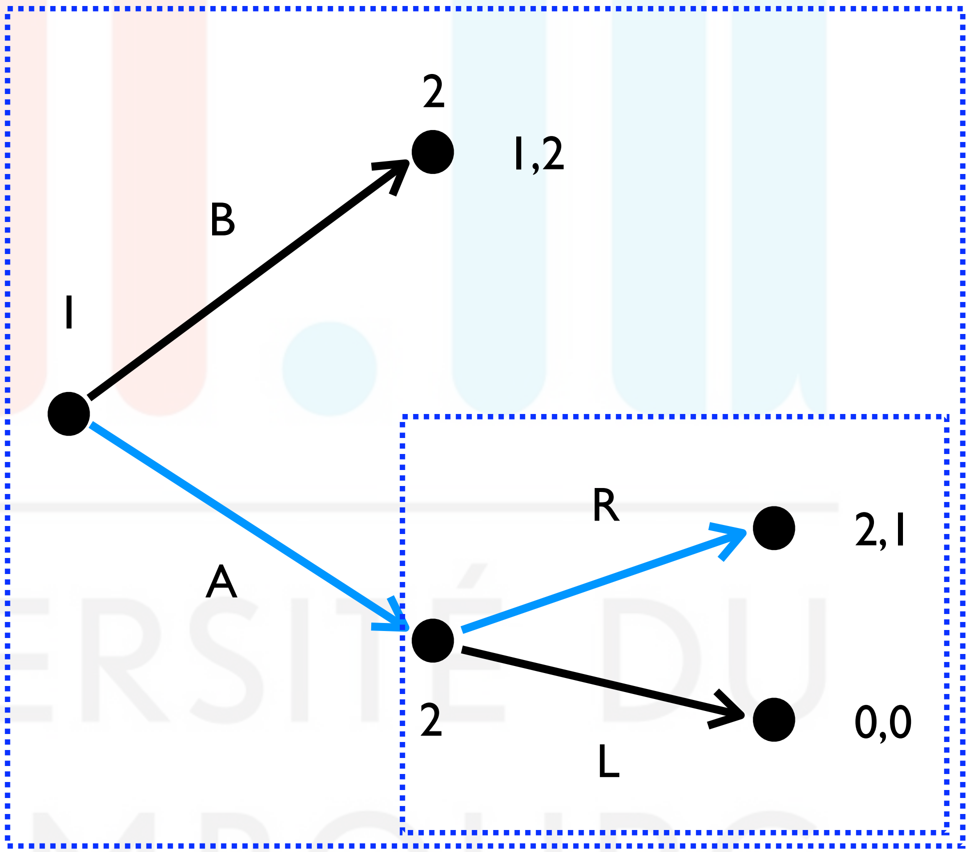
A simple BI



A simple BI



A simple BI



Extensions

Exogenous uncertainty (i)

Game $\Gamma = \langle N, H, P, f_c, (\precsim_i) \rangle$ is an extensive game with perfect information and chance moves if:

- N is a finite set of players
- H is the set of histories inductively defined as usual
- $P : H \backslash Z \longrightarrow N \cup \{c\}$
- $\forall h \in H$ s.t. $P(h) = c$, $f_c(\_|h)$ is a probability measure on $A(h)$. So $f_c(a|h)$ is the probability that a occurs after h
- $(\precsim_i)$ is a preference relation on probability distributions over Z

A strategy of i for Γ is defined as usual. The outcome of a strategy profile is a probability distribution over Z.

Subgame perfect equilibrium is defined as usual, i.e., an s^* s.t. $\forall i \in N$ and $h \in H \backslash Z$ for which $P(h) = i$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h O_h(s^*_{-i}|_h, s^*_i|_h)$$

$\forall s_i$ for subgame $\Gamma(h)$.

Exogenous uncertainty (i)

Game $\Gamma = \langle N, H, P, f_c, (\precsim_i) \rangle$ is an extensive game with perfect information and chance moves if:

- N is a finite set of players
- H is the set of histories inductively defined as usual
- $P : H \backslash Z \longrightarrow N \cup \{c\}$
- $\forall h \in H$ s.t. $P(h) = c$, $f_c(\_|h)$ is a probability measure on $A(h)$. So $f_c(a|h)$ is the probability that a occurs after h
- $(\precsim_i)$ is a preference relation on probability distributions over Z

A strategy of i for Γ is defined as usual. The outcome of a strategy profile is a probability distribution over Z.

Subgame perfect equilibrium is defined as usual, i.e., an s^* s.t. $\forall i \in N$ and $h \in H \backslash Z$ for which $P(h) = i$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h O_h(s^*_{-i}|_h, s^*_i|_h)$$

$\forall s_i$ for subgame $\Gamma(h)$.

Exogenous uncertainty (ii)

- The One Deviation Property and Kuhn's theorem hold for extensive games with perfect information and chance moves

Simultaneous moves (i)

Game $\Gamma = \langle N, H, P, (\precsim_i) \rangle$ is an extensive game with perfect information and simultaneous moves if:

- N is a finite set of players
- H is the set of histories inductively defined as usual
- $P : H \backslash Z \longrightarrow \mathcal{P}(N)$ and is s.t.: $\forall h \in H \backslash Z, \exists \{A_i(h)\}_{i \in P(h)}$ s.t. $A(h) = \{a \mid (h, a) \in H\} = \times_{i \in P(h)} A_i(h)$
- $(\precsim_i)$ are preference relations on Z

Histories are sequences of vectors whose components are the actions taken by the players whose turn it is to move.

A strategy of i is a function s_i associating to each $h \in H \backslash Z$ s.t. $P(h) = i$ an action in $A_i(h)$.

The definition of subgame perfect equilibrium remains the same (except for $P(h) = i$ being replaced by $i \in P(h)$).

An example

EXERCISE 103.1 Suppose that three players share a pie by using the following procedure. First player 1 proposes a division, then players 2 and 3 simultaneously respond either YES or NO. If players 2 and 3 both say YES then the division is implemented; otherwise no player receives anything. Each player prefers more of the pie to less. Formulate this situation as an extensive game with simultaneous moves and find its subgame perfect equilibria.

Simultaneous moves (ii)

- The One Deviation Property holds for extensive games with perfect information and simultaneous moves but Kuhn's theorem does not hold

Why no Kuhn? (exercise 103.3)

Let $\Gamma = \langle N, H, P, (\precsim_i) \rangle$ be a finite extensive game with perfect information. The subgame perferct equilibrium of Γ is built by induction on the length $l(\Gamma(h))$ of $\Gamma(h)$. For each length we define a function R associating a terminal history to every $h \in H$ and we show that such history is the outcome of the subgame perfect equilibrium of $\Gamma(h)$.

B: If $l(\Gamma(h)) = 0$, let $R(h) = h$

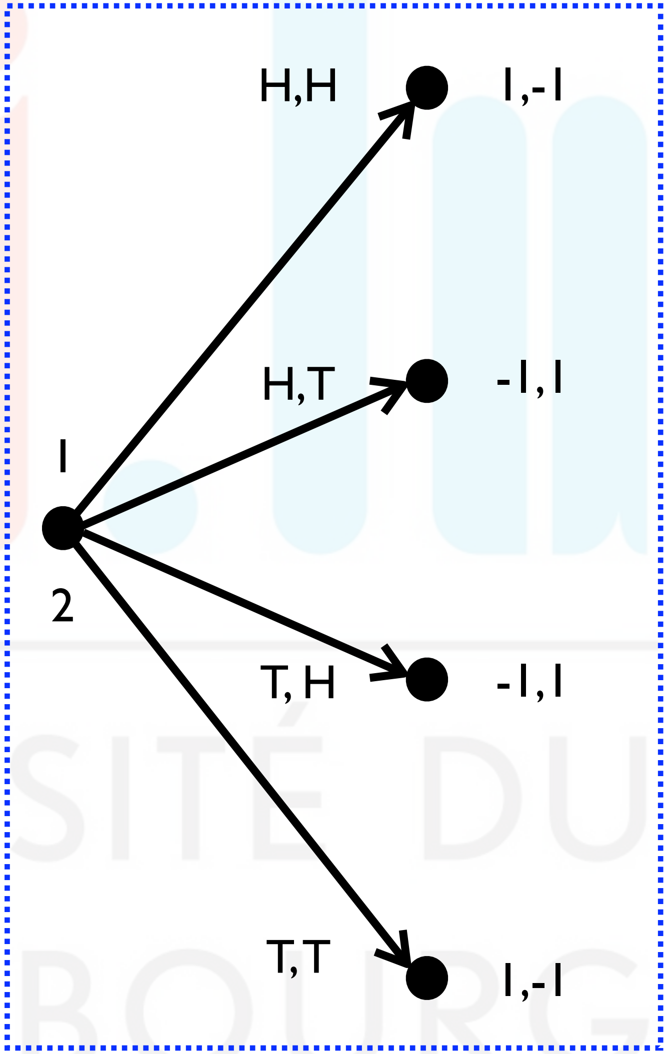
S: Suppose $R(h)$ is defined for all $h \in H$ such that $l(\Gamma(h)) \leq k$ with $0 \leq k$. Consider the history h^* such that $l(\Gamma(h^*)) = k+1$ and $P(h^*) = i$. Notice that for all $a \in A(h^*)$ $l(\Gamma(h^*, a)) \leq k$. Now $s_i(h^*)$ is defined to $\precsim_i$-maximize $R(h^*, a)$ over $a \in A(h^*)$ and $R(h^*) = R(h^*, s_i(h^*))$.

This constructs a strategy profile for Γ. Since it $\precsim_i$-maximizes the outcome over the set of actions immediately available after a given history, for the One Deviation Property, it is a subgame perfect equilibrium.

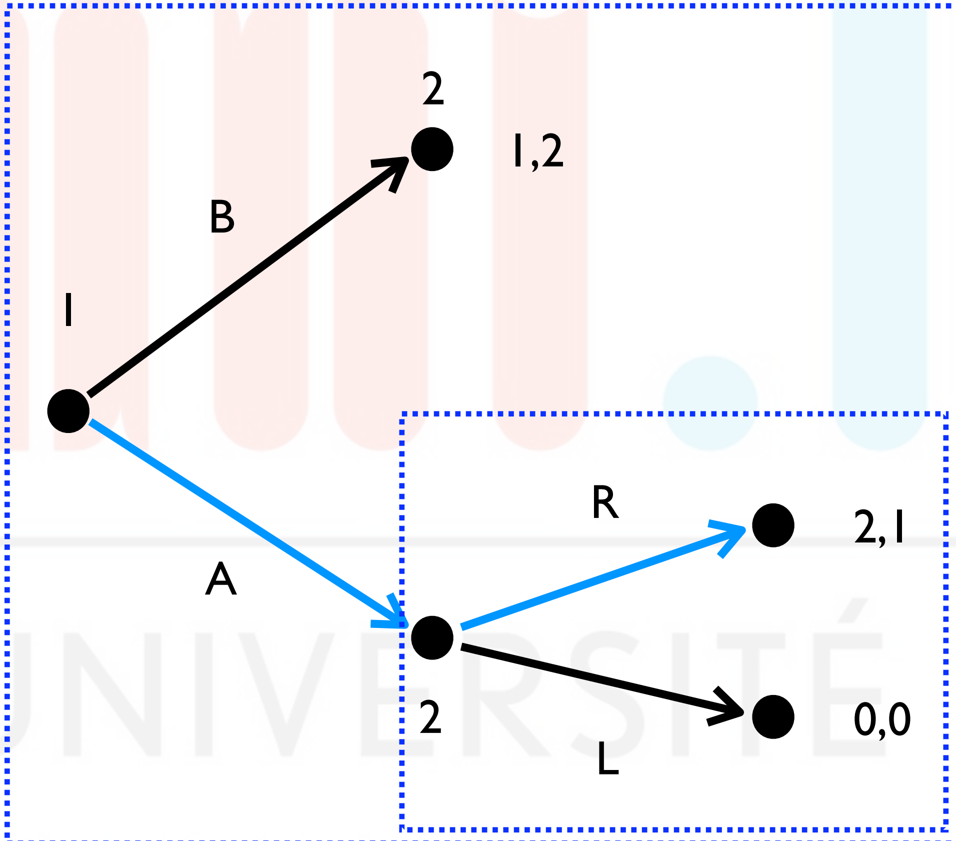
Why no Kuhn? Intuition

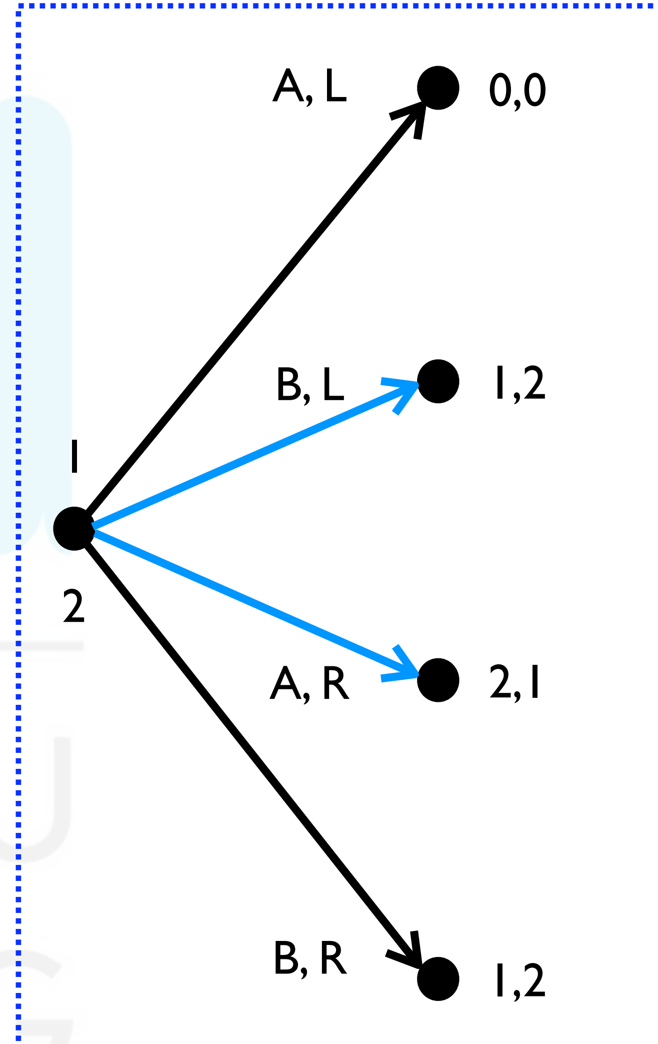
- We know that not all games with pure strategies have Nash Equilibria
- Hence it suffices to "translate" one of those games to extensive games with simultaneous actions to find a counterexample

Why no Kuhn? Counter-example



Nash vs. Subgame Perfect (i)





Back to strategic form!

Nash vs. Subgame Perfect (ii)

A Nash Equilibrium for $\Gamma = (N, (A_i), (\precsim_i))$ is a profile a^* s.t. $\forall i \in N$:

$$(a^*_{-i}, a_i) \precsim_i (a^*_{-i}, s^*_i)$$

$\forall a_i \in A_i$.

A Subgame Perfect Equilibrium for $\Gamma = \langle N, H, P, (\precsim_i)\rangle$ with simultaneous moves is a profile s^* s.t. $\forall i \in N$ and $h \in H \backslash Z$ for which $i = P(h)$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h\, O_h(s^*_{-i}|_h, s^*_i|_h)$$

$\forall s_i$ for subgame $\Gamma(h)$.

A Subgame Perfect Equilibrium for $\Gamma = \langle N, H, P, (\precsim_i)\rangle$ with simultaneous moves is a profile s^* s.t. $\forall i \in N$ and $h \in H \backslash Z$ for which $i \in P(h)$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h\, O_h(s^*_{-i}|_h, s^*_i|_h)$$

$\forall s_i$ for subgame $\Gamma(h)$.

Nash vs. Subgame Perfect (ii)

A Nash Equilibrium for $\Gamma = (N, (A_i), (\precsim_i))$ is a profile a^* s.t. $\forall i \in N$:

$$(a^*_{-i}, a_i) \precsim_i (a^*_{-i}, s^*_i)$$

$\forall a_i \in A_i$.

A Subgame Perfect Equilibrium for $\Gamma = \langle N, H, P, (\precsim_i) \rangle$ with simultaneous moves is a profile s^* s.t. $\forall i \in N$ and $h \in H \backslash Z$ for which $i = P(h)$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h O_h(s^*_{-i}|_h, s^*_i|_h)$$

$\forall s_i$ for subgame $\Gamma(h)$.

A Subgame Perfect Equilibrium for $\Gamma = \langle N, H, P, (\precsim_i) \rangle$ with simultaneous moves is a profile s^* s.t. $\forall i \in N$ and $h \in H \backslash Z$ for which $i \in P(h)$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h O_h(s^*_{-i}|_h, s^*_i|_h)$$

$\forall s_i$ for subgame $\Gamma(h)$.

Nash vs. Subgame Perfect (ii)

A Nash Equilibrium for $\Gamma = (N, (A_i), (\precsim_i))$ is a profile a^* s.t. $\forall i \in N$:

$$(a^*_{-i}, a_i) \precsim_i (a^*_{-i}, s^*_i)$$

$\forall a_i \in A_i$.

A Subgame Perfect Equilibrium for $\Gamma = \langle N, H, P, (\precsim_i)\rangle$ with simultaneous moves is a profile s^* s.t. $\forall i \in N$ and $h \in H \backslash Z$ for which $i = P(h)$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h O_h(s^*_{-i}|_h, s^*_i|_h)$$

$\forall s_i$ for subgame $\Gamma(h)$.

A Subgame Perfect Equilibrium for $\Gamma = \langle N, H, P, (\precsim_i)\rangle$ with simultaneous moves is a profile s^* s.t. $\forall i \in N$ and $h \in H \backslash Z$ for which $i \in P(h)$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h O_h(s^*_{-i}|_h, s^*_i|_h)$$

$\forall s_i$ for subgame $\Gamma(h)$.

Nash vs. Subgame Perfect (ii)

A Nash Equilibrium for $\Gamma = (N, (A_i), (\precsim_i))$ is a profile a^* s.t. $\forall i \in N$:

$$(a^*_{-i}, a_i) \precsim_i (a^*_{-i}, s^*_i)$$

$\forall a_i \in A_i$.

A Subgame Perfect Equilibrium for $\Gamma = \langle N, H, P, (\precsim_i)\rangle$ with simultaneous moves is a profile s^* s.t. $\forall i \in N$ and $h \in H \backslash Z$ for which $i = P(h)$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h O_h(s^*_{-i}|_h, s^*_i|_h)$$

$\forall s_i$ for subgame $\Gamma(h)$.

A Subgame Perfect Equilibrium for $\Gamma = \langle N, H, P, (\precsim_i)\rangle$ with simultaneous moves is a profile s^* s.t. $\forall i \in N$ and $h \in H \backslash Z$ for which $i \in P(h)$:

$$O_h(s^*_{-i}|_h, s_i) \precsim_i |_h O_h(s^*_{-i}|_h, s^*_i|_h)$$

$\forall s_i$ for subgame $\Gamma(h)$.

Iterated Elimination of Weakly Dominated Strategies

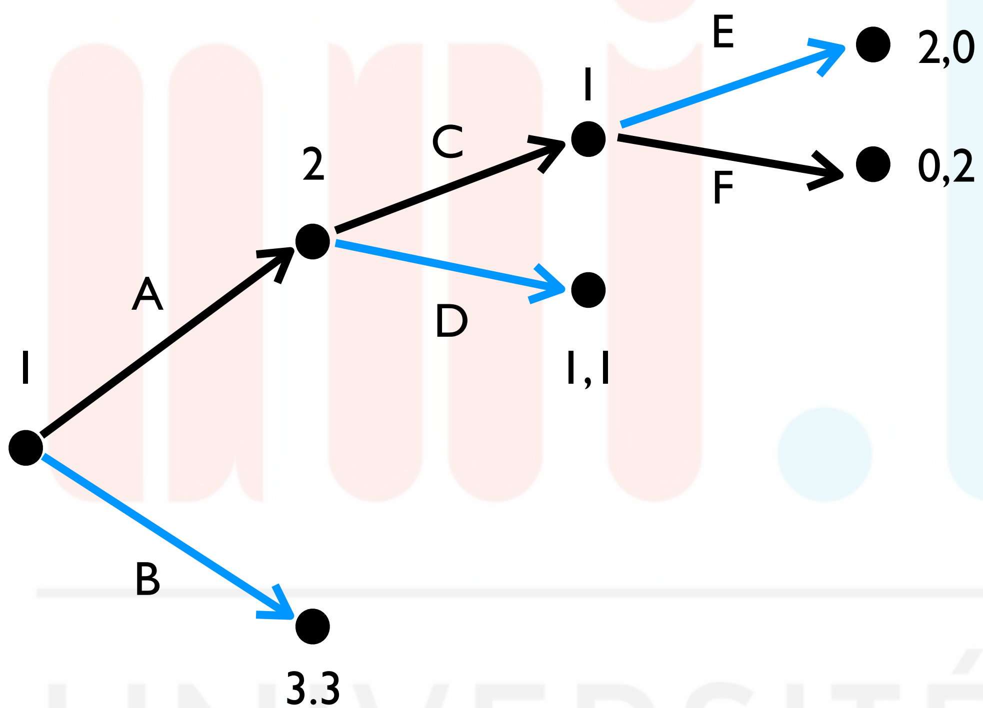
davide.grossi@uni.lu Individual and Collective Reasoning Group

UNIVERSITÉ DU LUXEMBOURG

Iterated elimination vs. Backwards Ind.

- The two procedures are related
- Given an extensive game with no indifference, it is possible to define a procedure for eliminating weakly dominated strategies in the strategic form of the same game, such that the remaining profiles generate the (unique) subgame perfect equilibrium of the extensive game
- Not strange! BI eliminates every strategy of a given player which differ for the action chosen after a given history

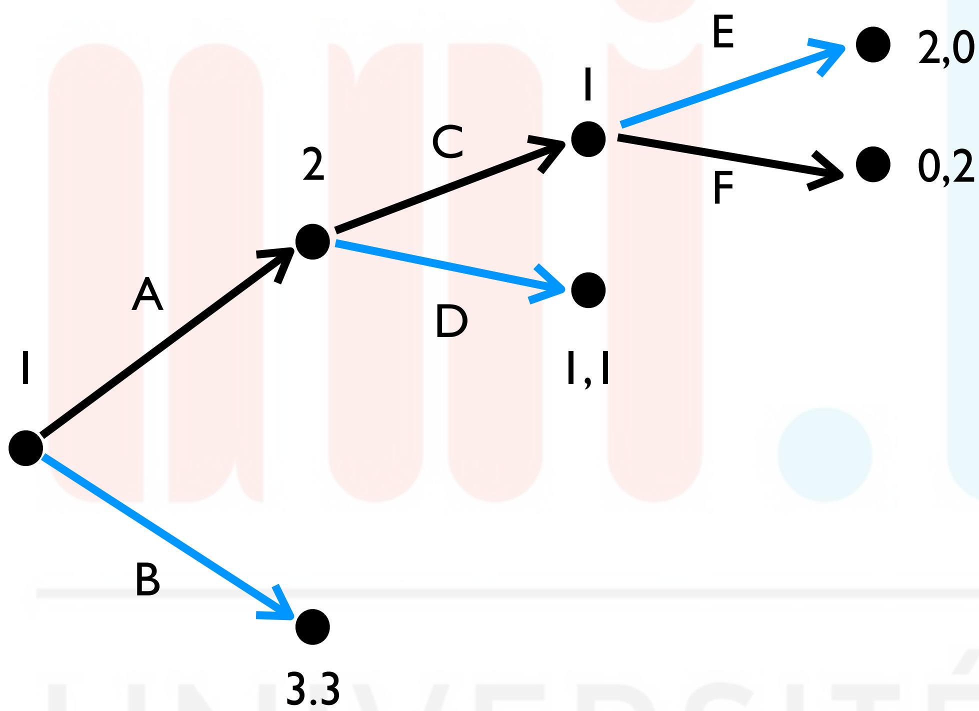
Example



1 \ 2	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

Every strategy of player i remaining at the end of the procedure chooses the actions selected by BI after any history that is consistent with player i's subgame perfect equilibrium strategy.

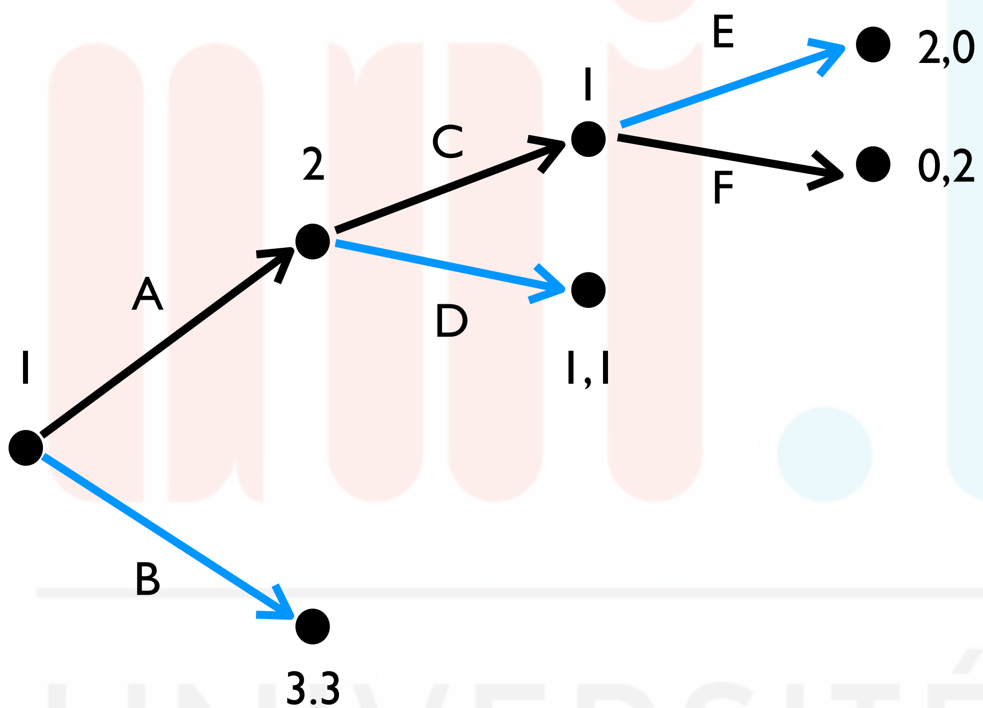
Example



1 \ 2	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

Every strategy of player i remaining at the end of the procedure chooses the actions selected by BI after any history that is consistent with player i's subgame perfect equilibrium strategy.

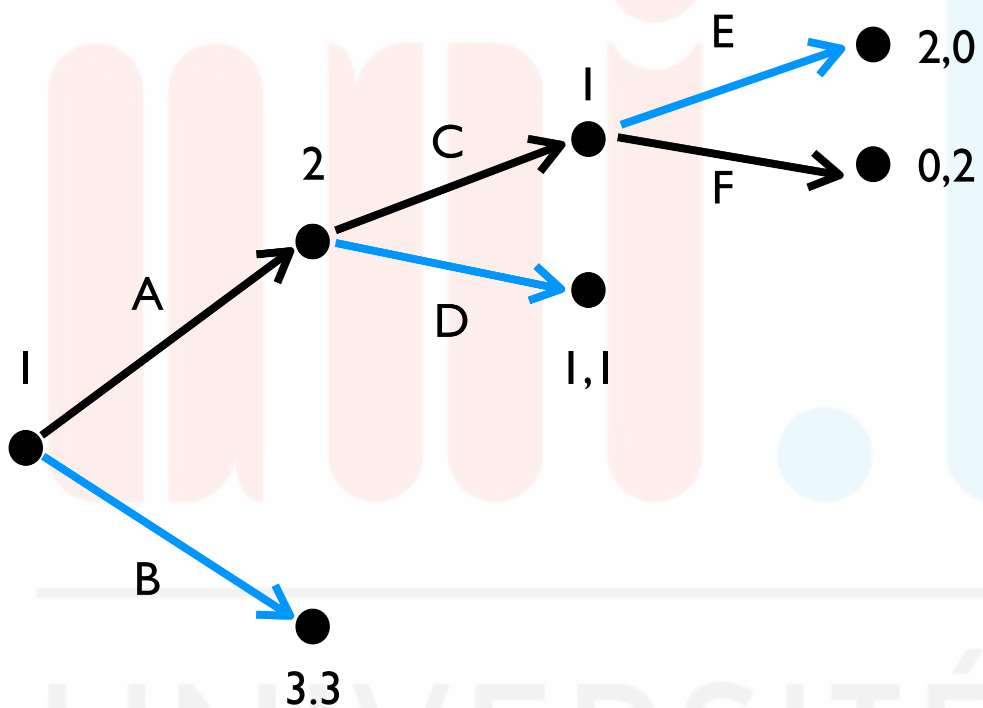
Example



1 \ 2	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

Every strategy of player i remaining at the end of the procedure chooses the actions selected by BI after any history that is consistent with player i's subgame perfect equilibrium strategy.

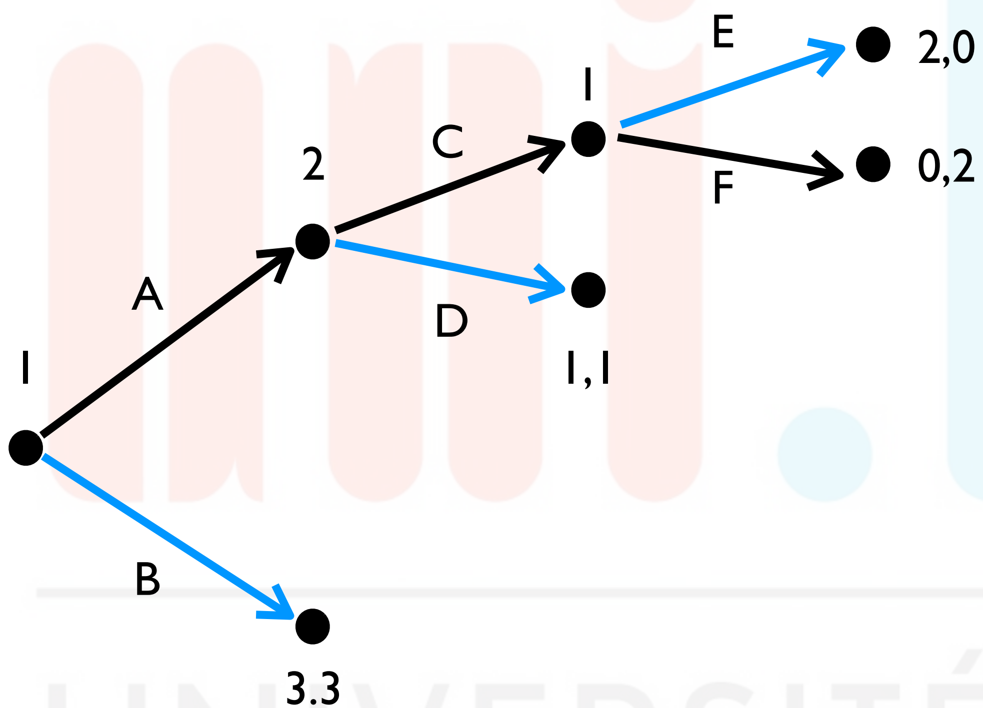
Example



1 \ 2	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

Every strategy of player i remaining at the end of the procedure chooses the actions selected by BI after any history that is consistent with player i's subgame perfect equilibrium strategy.

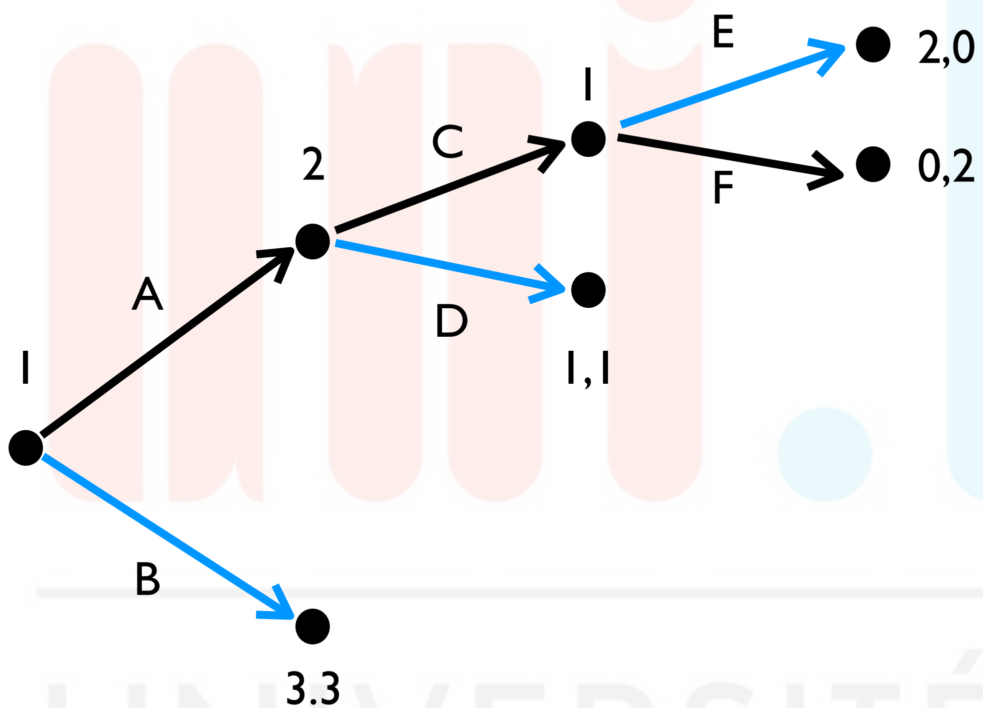
Example



1 \ 2	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

Every strategy of player i remaining at the end of the procedure chooses the actions selected by BI after any history that is consistent with player i's subgame perfect equilibrium strategy.

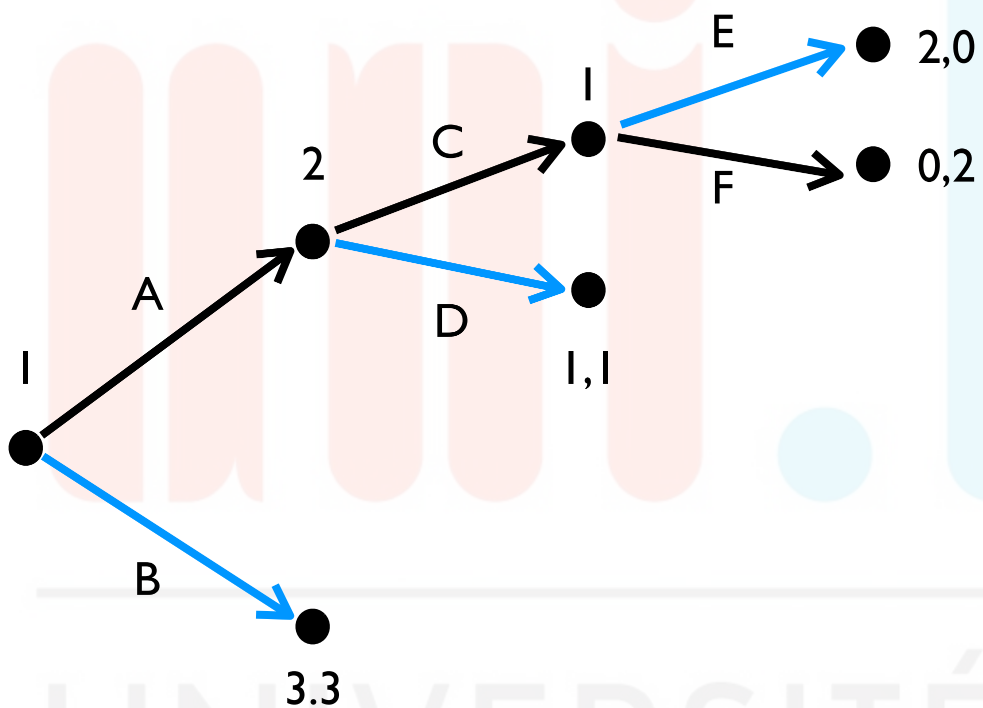
Importance of the order



1 \ 2	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

The order of elimination may end up removing all subgame perfect equilibria! Again, more information is encoded in extensive games!

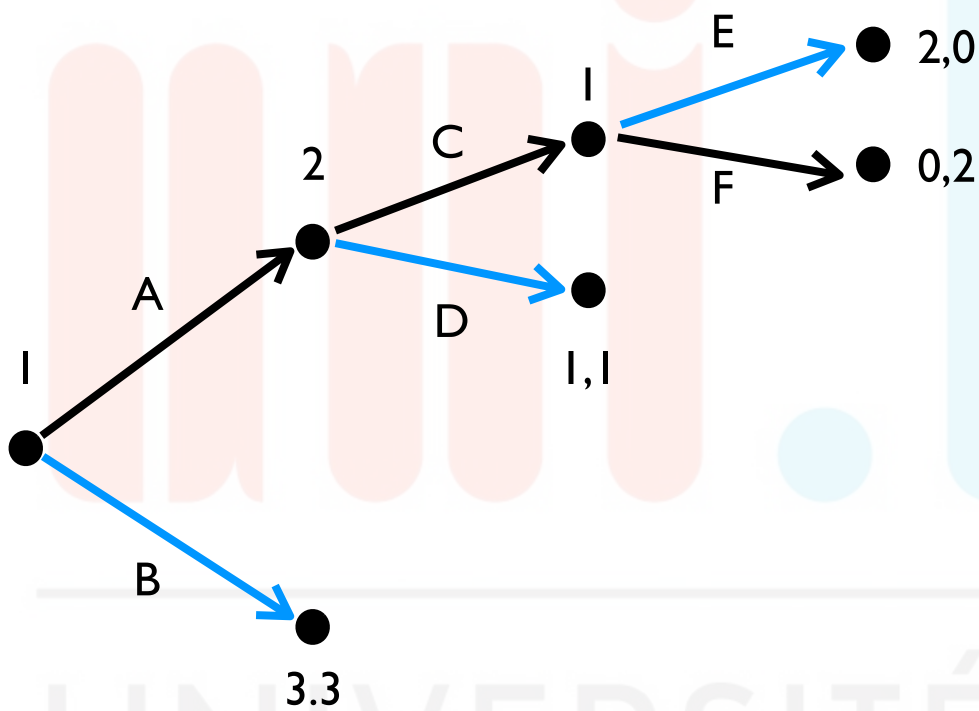
Importance of the order



1 \ 2	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

The order of elimination may end up removing all subgame perfect equilibria! Again, more information is encoded in extensive games!

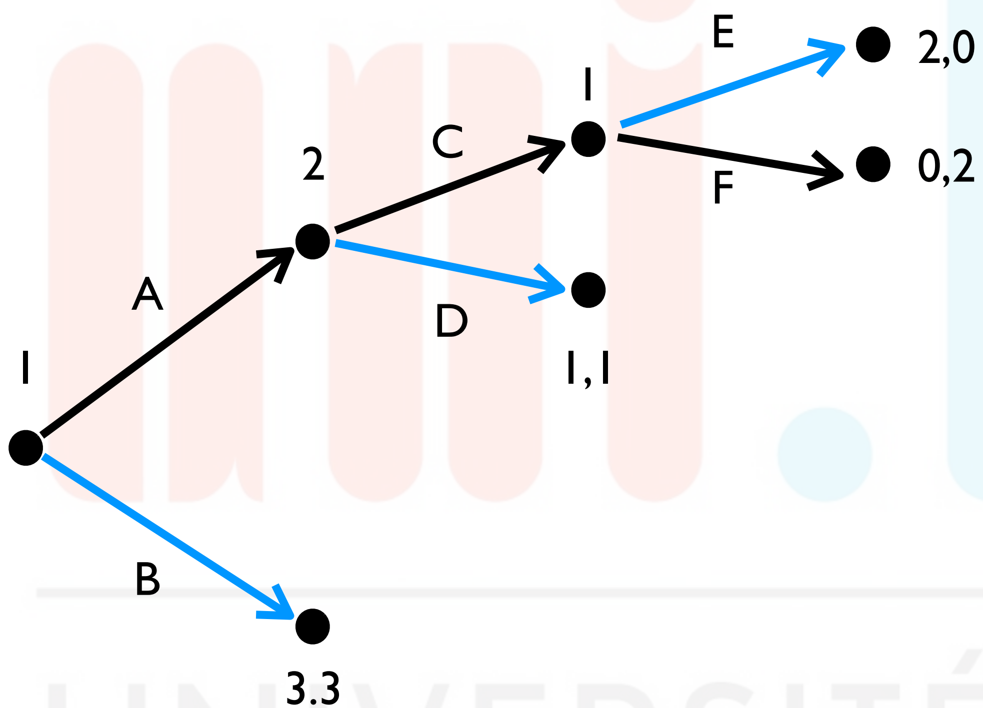
Importance of the order



1 \ 2	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

The order of elimination may end up removing all subgame perfect equilibria! Again, more information is encoded in extensive games!

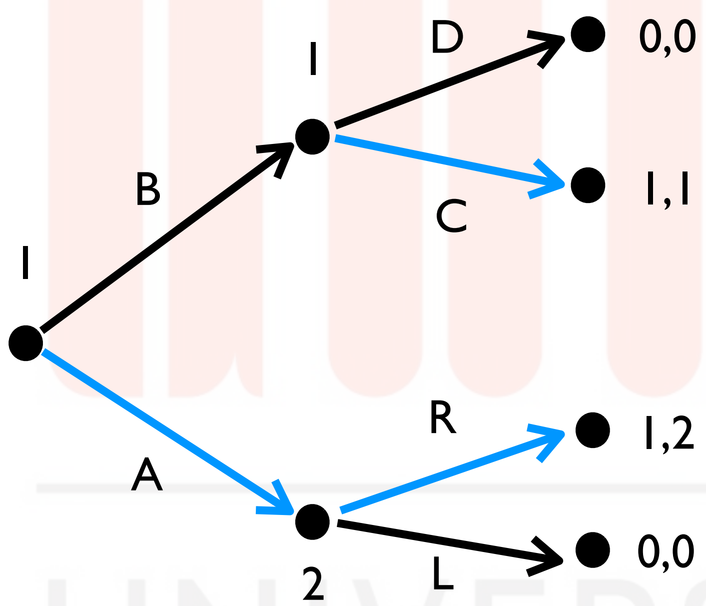
Importance of the order



1 \ 2	C	D
AE	2,0	1,1
AF	0,2	1,1
BE	3,3	3,3
BF	3,3	3,3

The order of elimination may end up removing all subgame perfect equilibria! Again, more information is encoded in extensive games!

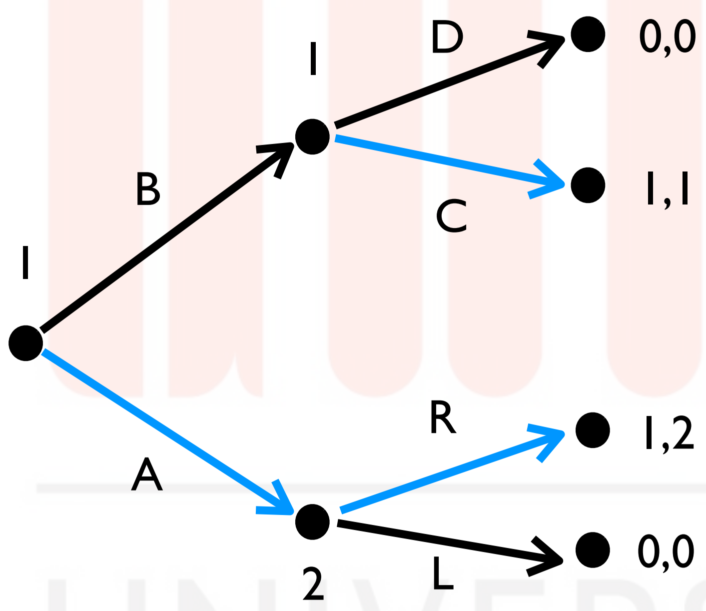
Effects of indifference (i)



1 \ 2	L	R
AC	0,0	1,2
AD	0,0	1,2
BC	1,1	1,1
BD	0,0	0,0

There is an order of elimination eliminating a SPE. In this case the one with oucome (A,R)

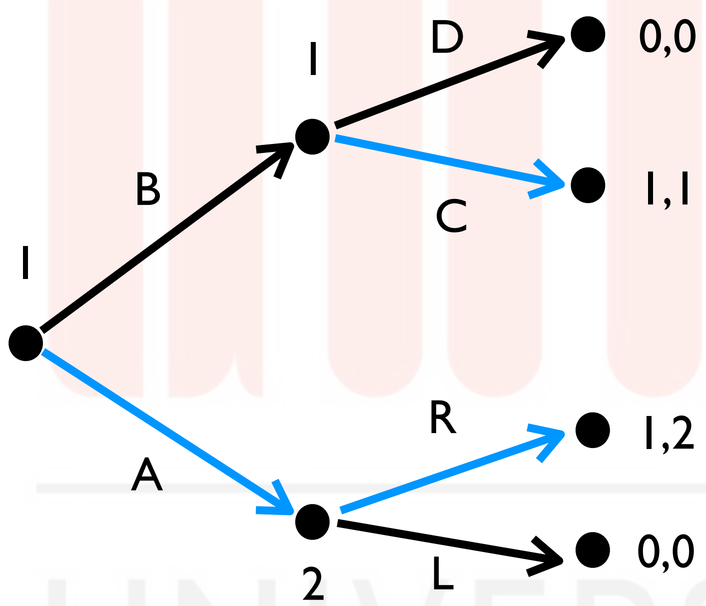
Effects of indifference (i)



1 \ 2	L	R
AC	0,0	1,2
AD	0,0	1,2
BC	1,1	1,1
BD	0,0	0,0

There is an order of elimination eliminating a SPE. In this case the one with oucome (A,R)

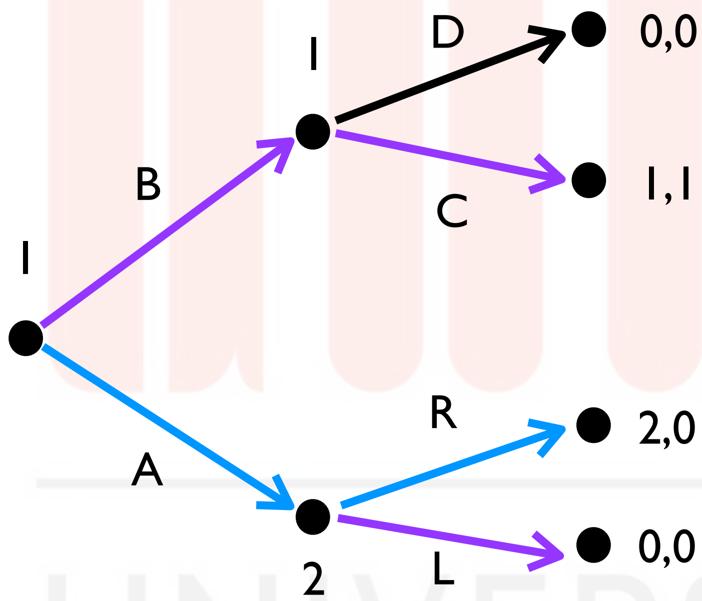
Effects of indifference (i)



1 \ 2	L	R
AC	0,0	1,2
AD	0,0	1,2
BC	1,1	1,1
BD	0,0	0,0

There is an order of elimination eliminating a SPE. In this case the one with oucome (A,R)

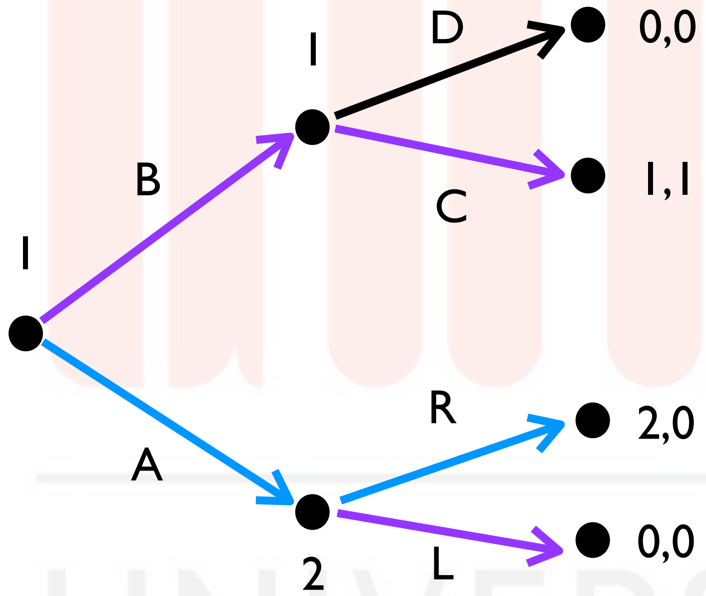
Effects of indifference (ii)



1 \ 2	L	R
AC	0,0	2,0
AD	0,0	2,0
BC	1,1	1,1
BD	0,0	0,0

There is no order of elimination s.t. all surviving profiles generate a SPE. In fact, outcome (A,L) is not even a Nash E., still it survives any order of elimination.

Effects of indifference (ii)

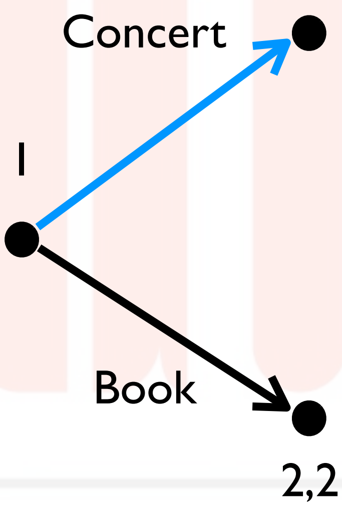


1 \ 2	L	R
AC	0,0	2,0
AD	0,0	2,0
BC	1,1	1,1
BD	0,0	0,0

There is no order of elimination s.t. all surviving profiles generate a SPE. In fact, outcome (A,L) is not even a Nash E., still it survives any order of elimination.

Forward induction

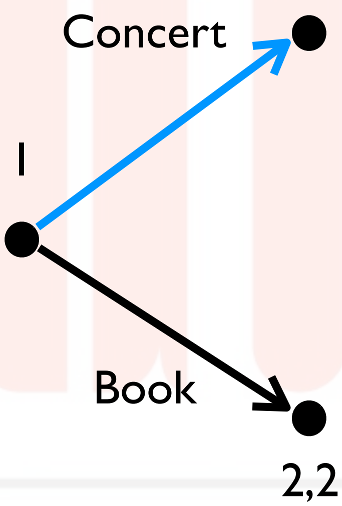
The Battle revisited (i)



1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	B	S
Book	2,2	2,2
B	3,1	0,0
S	0,0	1,3

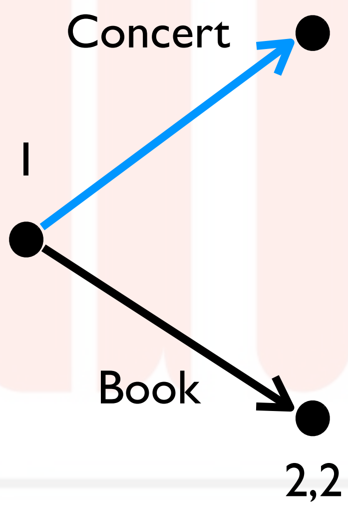
The Battle revisited (i)



1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	B	S
Book	2,2	2,2
B	3,1	0,0
S	0,0	1,3

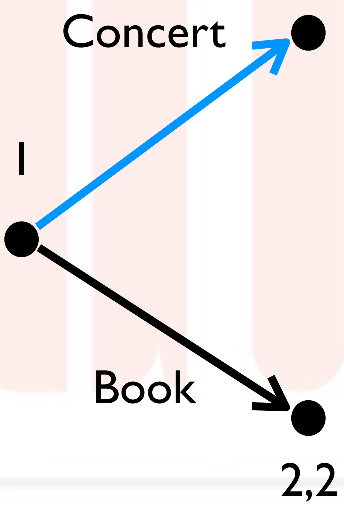
The Battle revisited (i)



1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	B	S
Book	2,2	2,2
B	3,1	0,0
S	0,0	1,3

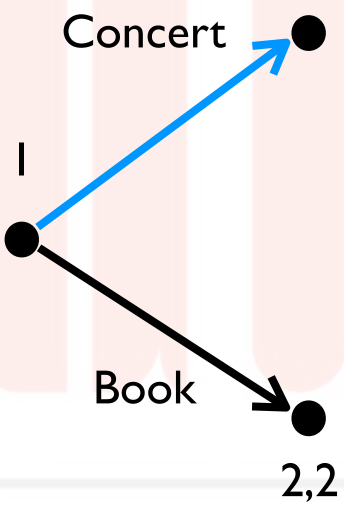
The Battle revisited (i)



1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	B	S
Book	2,2	2,2
B	3,1	0,0
S	0,0	1,3

The Battle revisited (i)

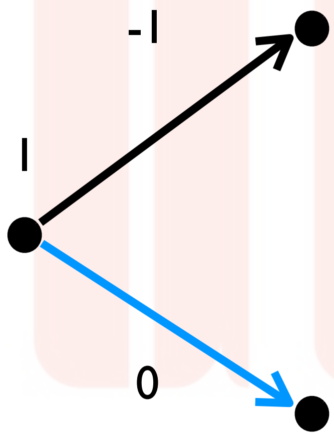


1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	B	S
Book	2,2	2,2
B	3,1	0,0
S	0,0	1,3

1. 2 knows that if she has to decide, then 1 has chosen concert
2. This makes sense for 1 only if she is going to choose B
3. Hence 2 chooses B too
4. *... but what about backwards induction?*

The Battle revisited (ii) or: "Burning money"

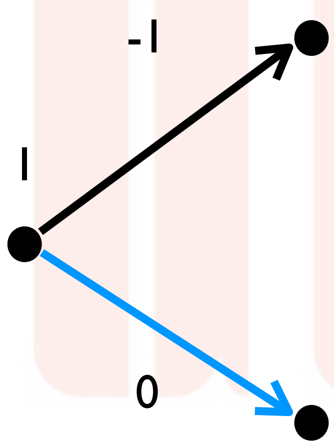


1 \ 2	B	S
B	2,1	-1,0
S	-1,0	0,3

1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	BB	BS	SB	SS
0B	3,1	3,1	0,0	0,0
0S	0,0	0,0	1,3	1,3
-1B	2,1	-1,0	2,1	-1,0
-1S	-1,0	0,3	-1,0	0,3

The Battle revisited (ii) or: “Burning money”

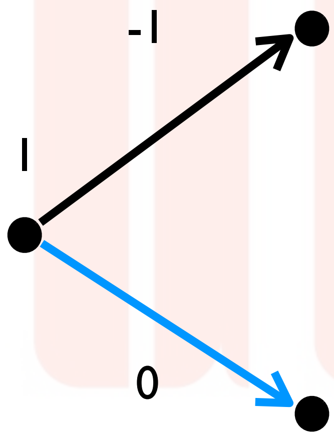


1 \ 2	B	S
B	2,1	-1,0
S	-1,0	0,3

1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	BB	BS	SB	SS
0B	3,1	3,1	0,0	0,0
0S	0,0	0,0	1,3	1,3
-1B	2,1	-1,0	2,1	-1,0
-1S	-1,0	0,3	-1,0	0,3

The Battle revisited (ii) or: "Burning money"

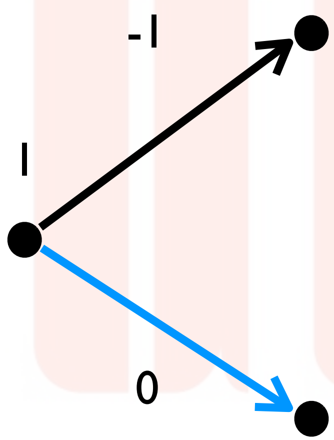


1 \ 2	B	S
B	2,1	-1,0
S	-1,0	0,3

1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	BB	BS	SB	SS
0B	3,1	3,1	0,0	0,0
0S	0,0	0,0	1,3	1,3
-1B	2,1	-1,0	2,1	-1,0
-1S	-1,0	0,3	-1,0	0,3

The Battle revisited (ii) or: "Burning money"

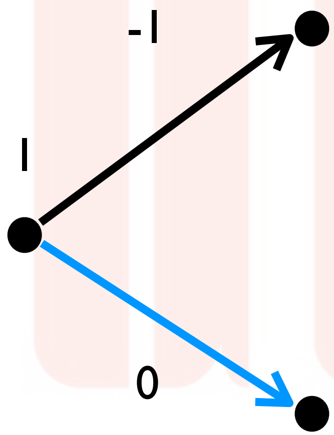


1 \ 2	B	S
B	2,1	-1,0
S	-1,0	0,3

1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	BB	BS	SB	SS
0B	3,1	3,1	0,0	0,0
0S	0,0	0,0	1,3	1,3
-1B	2,1	-1,0	2,1	-1,0
-1S	-1,0	0,3	-1,0	0,3

The Battle revisited (ii) or: “Burning money”



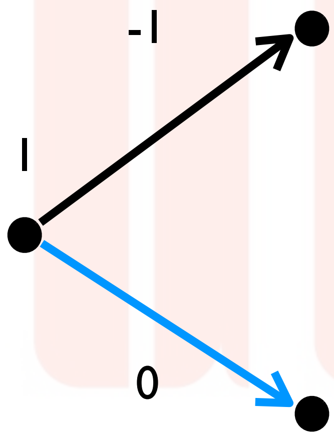
1 \ 2	B	S
B	2,1	-1,0
S	-1,0	0,3

1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	BB	BS	SB	SS
0B	3,1	3,1	0,0	0,0
0S	0,0	0,0	1,3	1,3
-1B	2,1	-1,0	2,1	-1,0
-1S	-1,0	0,3	-1,0	0,3

davide.grossi@uni.lu

The Battle revisited (ii) or: “Burning money”

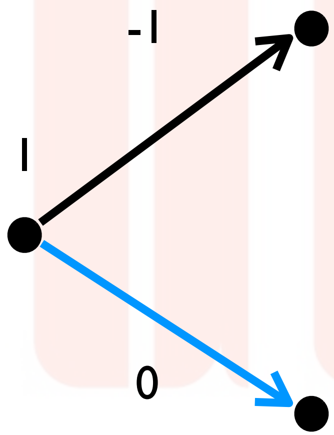


1 \ 2	B	S
B	2,1	-1,0
S	-1,0	0,3

1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	BB	BS	SB	SS
0B	3,1	3,1	0,0	0,0
0S	0,0	0,0	1,3	1,3
-1B	2,1	-1,0	2,1	-1,0
-1S	-1,0	0,3	-1,0	0,3

The Battle revisited (ii) or: "Burning money"

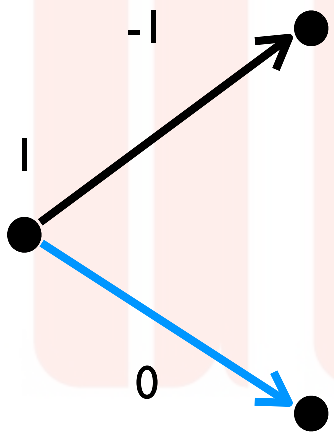


1 \ 2	B	S
B	2,1	-1,0
S	-1,0	0,3

1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	BB	BS	SB	SS
0B	3,1	3,1	0,0	0,0
0S	0,0	0,0	1,3	1,3
-1B	2,1	-1,0	2,1	-1,0
-1S	-1,0	0,3	-1,0	0,3

The Battle revisited (ii) or: "Burning money"



1 \ 2	B	S
B	2,1	-1,0
S	-1,0	0,3

1 \ 2	B	S
B	3,1	0,0
S	0,0	1,3

1 \ 2	BB	BS	SB	SS
0B	3,1	3,1	0,0	0,0
0S	0,0	0,0	1,3	1,3
-1B	2,1	-1,0	2,1	-1,0
-1S	-1,0	0,3	-1,0	0,3

1. The mere possibility for 1 to throw away 1 EUR is sufficient for yielding 1's preferred outcome!

2. These games model a form of message exchange