Game Theory Seminar

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Schedule

- 2.5 Elimination of dominated strategies
- 2.6 Multiagent representation

Domination in decision theory

Given the utility function $u : X \times \Omega \rightarrow R$, a decision option y in X is strongly dominated by a randomized strategy σ in $\Delta(X)$ such that

$$\sum_{x \in X} \sigma(x) u(x,t) > u(y,t), \qquad \forall t \in \Omega.$$

That is, y is strongly dominated by σ if, no matter what the state might be, σ would always be strictly better than y.

A randomized strategy σ is any probability distribution over the set of decision options X.

Example

$X = \{\alpha, \beta, \gamma\}, \Omega = \{\theta_1, \theta_2\}$

Decision	θ_1	θ_2
α	8	1
β	5	3
γ	4	7

β is strongly dominated by the randomized strategy $0.5[\alpha] + 0.5[\gamma]$.

- If the true state were θ_1, then $0.5 \times 8 + 0.5 \times 4 = 6 > 5$.
- If the true state were θ_2, then $0.5 \times 1 + 0.5 \times 7 = 4 > 3$.

Strongly dominated strategy

Given any strategic-form game $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, any player i in N, and any strategy d_i in C_i, d_i is strongly dominated for player i iff there exists some randomized strategy σ_i in $\Delta(C_i)$ such that

$$\sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) > u_i(c_{-i}, d_i), \qquad \forall c_{-i} \in C_{-i}.$$

A randomized strategy σ for a player i is any probability distribution over the set of C_i.

Example

The simple card game in the strategic form.

	C_2	
C_1	M	P
Rr	0, 0	1, -1
Rf	0.5, -0.5	0, 0
Fr	-0.5, 0.5	1, -1
Ff	0, 0	0, 0

The strategy Ff is strongly dominated for player 1 by the randomized strategy $0.5[Rr] + 0.5[Rf] + 0.0[Fr]$.

Elimination of strongly dominated strategy

d_i is strongly dominated for player i if and only if d_i can never be the best response for i, no matter what he may believe about the other players' strategies.

So, eliminating a strongly dominated strategy for any player i should affect the analysis of the game, because player i would never use this strategy, and this fact should be evident to the other players if they are intelligent.

After one or more strongly dominated strategies have been eliminated from a game, other strategies that were not strongly dominated in the original game may become strongly dominated in the game that remains.

Example

A game in the strategic form.

	C_2		
C_1	x_2	y_2	z_2
a_1	$2, 3$	$3, 0$	$0, 1$
b_1	$0, 0$	$1, 6$	$4, 2$

Question: for player 2, which strategy is strongly dominated?

Example

A game in the strategic form.

	C_2		
C_1	x_2	y_2	z_2
a_1	$2,$ 3	$3,$ 0	$0,$ 1
b_1	$0,$ 0	$1,$ 6	$4,$ 2

z_2 is strongly dominated for player 2 by $0.5[x_2] + 0.5[y_2]$.

Example

A game in the strategic form.

	C_2	
C_1	x_2	y_2
a_1	$2, 3$	$3, 0$
b_1	$0, 0$	$1, 6$

Question: for player 1, which strategy is strongly dominated?

Example

A game in the strategic form.

	C_2	
C_1	x_2	y_2
a_1	2, 3	3, 0
b_1	0, 0	1, 6

b_1 is strongly dominated for player 1.

Example

A game in the strategic form.

	C_2	
C_1	x_2	y_2
a_1	$2, 3$	$3, 0$

Question: for player 2, which strategy is strongly dominated?

Example

A game in the strategic form.

	C_2	
C_1	x_2	y_2
a_1	$2,$ 3	$3,$ 0

y_2 is strongly dominated for player 2.

Example

A game in the strategic form.

	C_2
C_1	x_2
a_1	$2, 3$

In this game, the iterative elimination of strongly dominated strategy leads to a unique prediction as to what the players should do.

Elimination process

Given a general strategic form game $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$. For any player i, let $C_i^{(1)}$ denote the set of all strategies in C_i that are not strongly dominated for i. Then let $\Gamma^{(1)}$ be the strategic form game

$$\Gamma^{(1)} = (N, (C_i^{(1)})_{i \in N}, (u_i)_{i \in N}).$$

Then by induction, for every positive integer k, we can define the strategic form game $\Gamma^{(k)}$.

$$\Gamma^{(k)} = (N, (C_i^{(k)})_{i \in N}, (u_i)_{i \in N})$$

where for each player i, $C_i^{(k)}$ is the set of all strategies in $C_i^{(k-1)}$ that are not strongly dominated for i in the game $\Gamma^{(k-1)}$.

Note that the utility function has to be reinterpreted to smaller domain $\times_{j \in N} C_j^{(k)}$.

Elimination process

It is easy to see that

$$C_i \supseteq C_i^{(1)} \supseteq C_i^{(2)} \supseteq C_i^{(3)} \supseteq \cdots$$

Since we started with a finite game Γ, there must exist some number K such that

$$C_i^{(K)} = C_i^{(K+1)} = C_i^{(K+2)} = \cdots, \quad \forall i \in N$$

Let $\Gamma^{(\infty)} = \Gamma^{(K)}$ and $C_i^{(\infty)} = C_i^{(K)}$ for every $i \in N$. The strategies in $C_i^{(\infty)}$ are iteratively undominated. The game $\Gamma^{(\infty)}$ is the residual game generated from Γ by iterative strong domination.

Weakly dominated strategy

Given any strategic-form game $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, any player i in N, and any strategy d_i in C_i, d_i is weakly dominated for player i iff there exists some randomized strategy σ_i in $\Delta(C_i)$ such that

$$\sum_{e_i \in C_i} \sigma_i(e_i) u_i(c_{-i}, e_i) \geq u_i(c_{-i}, d_i), \qquad \forall c_{-i} \in C_{-i},$$

and, for at least one strategy combination $\hat{c}_{-i}$ in C_{-i},

$$\sum_{e_i \in C_i} \sigma_i(e_i) u_i(\hat{c}_{-i}, e_i) > u_i(\hat{c}_{-i}, d_i).$$

Question: can we eliminate weakly dominated strategies?

Elimination of weakly dominated strategy

It is harder to argue that eliminating a weakly dominated strategy should not affect the analysis of games, because weakly dominated strategies could be the best responses for a player, if he feels confident that some strategies of other players have probability 0.

Theorem 1.7: Given the utility function $u : X \times \Omega \rightarrow R$, given y in X, there exist a randomized strategy σ in $\Delta(X)$ such that y is weakly dominated by σ if and only if there does not exist any probability distribution p in $\Delta^{>0}(\Omega)$ such that y is optimal.

$\Delta^{>0}(\Omega)$ is the set of probability distribution that assign strictly positive probability to every state in Ω.

More about technical difficulties ...

Example

A game in the strategic form.

	C_2	
C_1	x_2	y_2
x_1	3, 2	2, 2
y_1	1, 1	0, 0
z_1	0, 0	1, 1

z_1 is strongly dominated for player 1.

Example

A game in the strategic form.

	C_2	
C_1	x_2	y_2
x_1	$3, 2$	$2, 2$
y_1	$1, 1$	$0, 0$

Question: which strategy is weakly dominated for player 2?

A game in the strategic form.

	C_2	
C_1	x_2	y_2
x_1	$3,$ 2	$2,$ 2
y_1	$1,$ 1	$0,$ 0

y_2 is weakly dominated for player 2.

Example

A game in the strategic form.

	C_2	
C_1	x_2	y_2
x_1	3, 2	2, 2
y_1	1, 1	0, 0
z_1	0, 0	1, 1

y_1 is strongly dominated for player 1.

Example

A game in the strategic form.

	C_2	
C_1	x_2	y_2
x_1	$3, 2$	$2, 2$
z_1	$0, 0$	$1, 1$

Question: which strategy is weakly dominated for player 2?

Example

A game in the strategic form.

	C_2	
C_1	x_2	y_2
x_1	$3,$ 2	$2,$ 2
z_1	$0,$ 0	$1,$ 1

x_2 is weakly dominated for player 2.

Example

A game in the strategic form.

	C_2	
C_1	x_2	y_2
x_1	3 , 2	2 , 2
y_1	1 , 1	0 , 0
z_1	0 , 0	1 , 1

y_1 and z_1 are strongly dominated for player 1.

Example

A game in the strategic form.

C_1	C_2: x_2	C_2: y_2
x_1	$3, 2$	$2, 2$

Question: which strategy is weakly dominated for player 2?

Example

A game in the strategic form.

	C_2	
C_1	x_2	y_2
x_1	$3, 2$	$2, 2$

Neither of player 2's strategies would be weakly dominated!

Thus, which of player 2's strategies would be eliminated by a process of iterative elimination of weakly dominated strategies depends on the order in which we eliminate player 1's dominated strategies.

Elimination of weakly dominated strategy

Eliminating strategies for other players can never cause a strongly dominated strategy for player i to cease being strongly dominated, but it can cause a weakly dominated strategy to cease being weakly dominated.

Elimination of weakly dominated strategy

They are still useful concepts!

The simple card game in the strategic form.

	C_2	
C_1	M	P
Rr	0 $, 0$	1 $, -1$
Rf	0.5 $, -0.5$	0 $, 0$
Fr	-0.5 $, 0.5$	1 $, -1$
Ff	0 $, 0$	0 $, 0$

Fr and Ff are weakly dominated strategies for player 1.

It expresses that the intuition that player 1 should not fold when he has a winning card.

Motivation for multiagent representation

It should not matter if a given player in a game Γ^e were represented by a different agent in each of his possible information states, provided that these agents all share the same preferences and information of the original player.

Multiagent representation

Let Γ^e be any given game in extensive form, and let N denote the set of players in Γ^e. For any $i \in N$, let S_i denote the set of information states for player i that occur at the various nodes belonging to i in the game. We assume that these S_i sets are disjoint, $S_i \cap S_j = \emptyset$ if $i \neq j$.

The set of players in the multiagent representation of Γ^e is

$$S^* = \cup_{i \in N} S_i.$$

The players in the multiagent representation are referred as temporary agents.

A temporary agent r representing player i is responsible for choosing the move that i would make when the path of play reaches a node that is controlled by i with the information state r.

Multiagent representation

Let D_r be the set of moves controlled by player i in the information state r. (It denotes the set of strategies for the temporary agent r in the multiagent representation of Γ^e.)

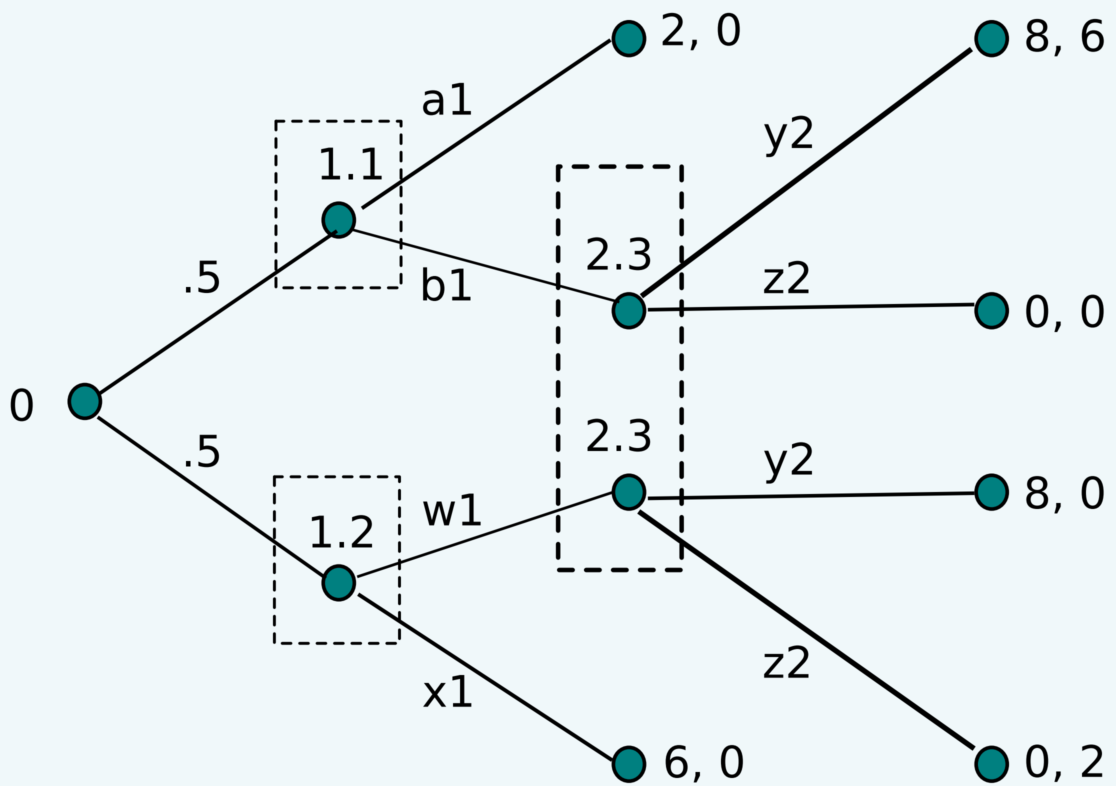
The utility functions v_r for the temporary agents are defined to coincide with the utility functions u_i of the corresponding players in the normal representation.

Formally, for any $i \in N$ and any $r \in S_i$, we define $v_r : \text{X}_{s \in S^*} D_s \rightarrow R$, so that for any $(d_s)_{s \in S^*}$ in $\text{X}_{s \in S^*} D_s$, if $(c_j)_{j \in N}$ is the strategy profile for the normal representation such that $c_j(t) = d_t$ for every $j \in N$ and every t in S_j, then $v_r((d_s)_{s \in S^*}) = u_i((c_j)_{j \in N})$.

Now, we arrive at the multiagent representation of Γ^e:

$$(S^*, (D_r)_{r \in S^*}, (v_r)_{r \in S^*}).$$

Example



Question: can you construct the normal representation of the game?

Example

$N = \{1, 2\}, C_1 = \{a_1w_1, a_1x_1, b_1w_1, b_1x_1\}, C_2 = \{y_2, z_2\}$

	C_2	
C_1	y_2	z_2
a_1w_1	$5, 0$	$1, 1$
a_1x_1	$4, 0$	$4, 0$
b_1w_1	$8, 3$	$0, 1$
b_1x_1	$7, 3$	$3, 0$

Question: which strategy is strongly dominated?

Example

$$N = \{1, 2\}, C_1 = \{a_1w_1, a_1x_1, b_1w_1, b_1, x_1\}, C_2 = \{y_2, z_2\}$$

	C_2	
C_1	y_2	z_2
a_1w_1	5, 0	1, 1
a_1x_1	4, 0	4, 0
b_1w_1	8, 3	0, 1
b_1x_1	7, 3	3, 0

a_1w_1 is strongly dominated for player i.

Example

The corresponding multiagent representation:

$$S^* = \{1, 2, 3\}, D_1 = \{a_1, b_1\}, D_2 = \{w_1, x_1\}, D_3 = \{y_2, z_2\}$$

	y_2		z_2	
	w_1	x_1	w_1	x_1
a_1	$5, 5, 0$	$4, 4, 0$	$1, 1, 1$	$4, 4, 0$
b_1	$8, 8, 3$	$7, 7, 3$	$0, 0, 1$	$3, 3, 0$

Question: which strategy is strongly dominated?

Example

The corresponding multiagent representation:

$$S^* = \{1, 2, 3\}, D_1 = \{a_1, b_1\}, D_2 = \{w_1, x_1\}, D_3 = \{y_2, z_2\}$$

	y_2		z_2	
	w_1	x_1	w_1	x_1
a_1	$5, 5, 0$	$4, 4, 0$	$1, 1, 1$	$4, 4, 0$
b_1	$8, 8, 3$	$7, 7, 3$	$0, 0, 1$	$3, 3, 0$

No strategies are strongly or weakly dominated. (For each temporary agent, each of his two strategies is a unique best response to some combination of strategies by the other two agents.)

Conclusion

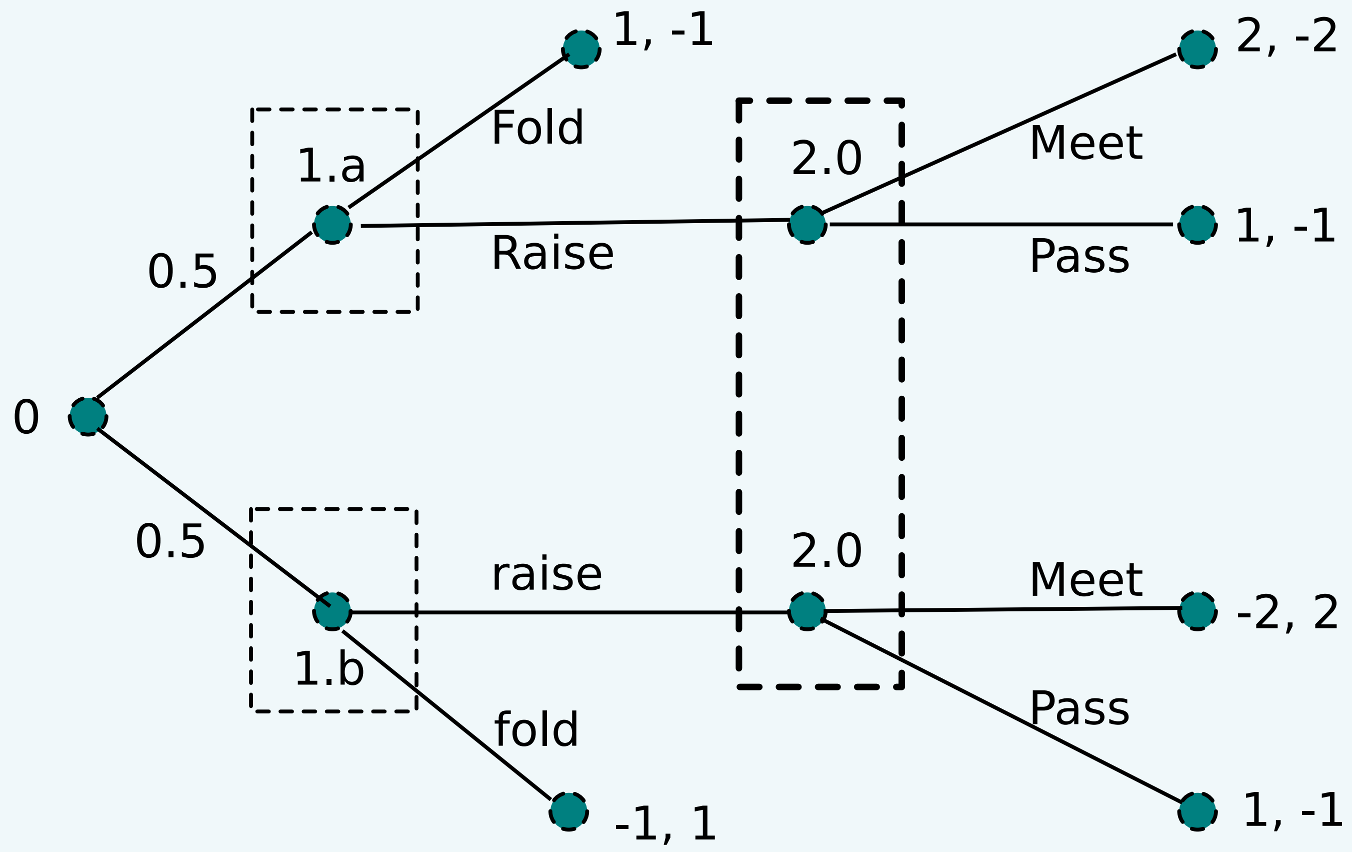
A domination argument that may seem rather convincing when we only consider the normal representation becomes more questionable when we consider the multiagent representation.

Schedule

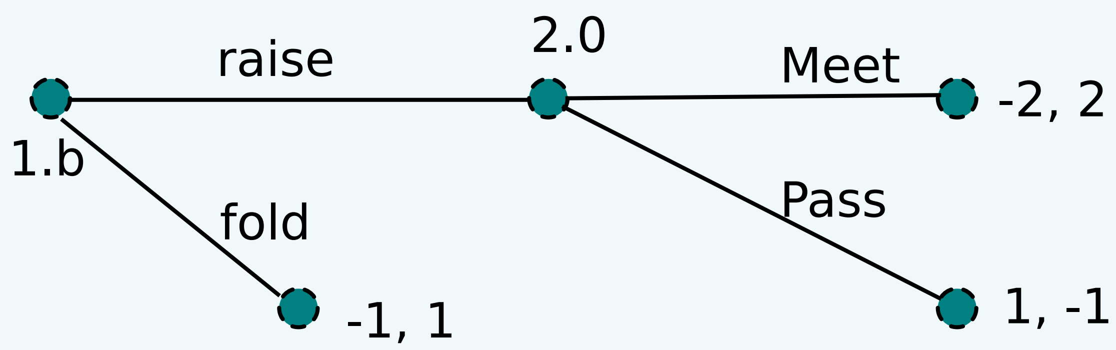
- 2.7 Common knowledge
- 2.8 Bayesian games
- 2.9 Modelling with incomplete information

Example

The simple card game:

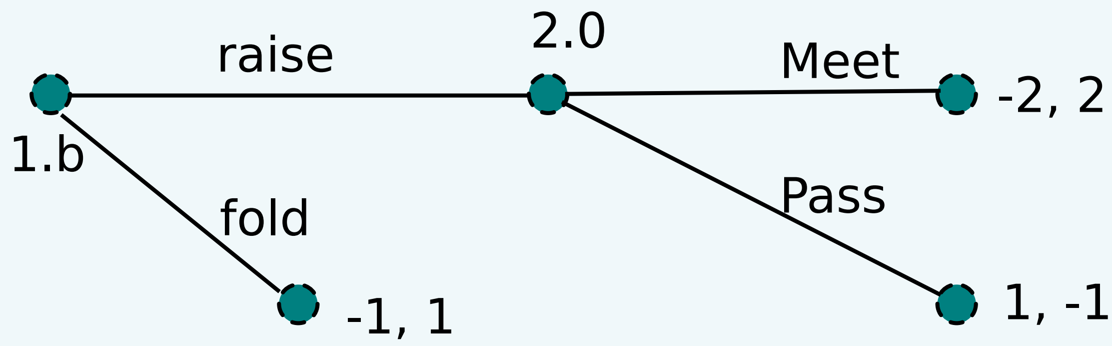


Example



Player 1 has drawn a black card, he intends to reason as modeled in the above figure.

Example



Player 1 has drawn a black card, he intends to reason as modeled in the above figure.

However, player 2 does know payoffs as shown above. She does not know the color of the card.

Common knowledge

A fact is common knowledge among the players if every player knows it, every player knows that every player knows it, and so on.

So common knowledge is a statement of the form "(every player knows that)^k every player knows it" is true, for $k = 0, 1, 2, \cdots$.

A player's private information is any information that he has that is not common knowledge among all the players in the game.

Common knowledge

A model of game must be common knowledge among the players (by the intelligence assumption):

- whatever we know or understand about the game must be known or understood by the players of the game, since they are as intelligent as we are;

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Common knowledge

A model of game must be common knowledge among the players (by the intelligence assumption):

- whatever we know or understand about the game must be known or understood by the players of the game, since they are as intelligent as we are;
- whatever model of the game we may study, we must assume that the players know this model too;
- furthermore, since we know that the players all know the model, the intelligent players must know that they all know the model;
- now, we also recognize that the intelligent players also know that they all know that they all know the model;
- and so on ...

Common knowledge

A node is called historical node, at the time when the game model is formulated and analyzed, the outcome of this node has already occurred and is known to some (but not all) players.

The root node in the extensive form must represent a situation at some time in the past before the players learned their private information, so everything that any player then knew about the game was common knowledge.

All relevant private information that players may have now must be accounted for by nodes and branches representing the past events that the players may have observed.

The cheating husbands puzzle

A certain village contains a number of married couples, of which 100 husbands are cheating on their wives. Every woman is aware of all the cheating taking place, expect for the infidelities of her own husband. In order to uphold a strict morality, the women of the village make a pact: any woman who learns that her husband has been cheating will bring her husband to the public square of the town for all to see. However, because no one wants to tell another woman that her husband is being unfaithful, this information is never communicated, and so, the cheating continues.

Some time later, at a town meeting, the chief announces, “I want the cheating in this village to stop." Then, 99 days pass uneventfully, but on the 100-th day, all 100 cheating husbands are put at the public square by their wives.

Question: how did this happen?

The cheating husbands puzzle

- If there is only one cheating husband, his wife knows that no one else's husband is cheating, so hers must be because there is at least one cheating husband. She brings him to the public square on day one.
- If there are two cheating husbands, each wife sees the other cheated-on wife not bring her husband to the public square on day one, and therefore concludes that her own husband is cheating. (The other wife would only bring her husband to the public square if she knew that no other husbands were cheating. Because she doesn't bring him, at least one other husband must be cheating. Because the only unknown value for the first wife is her own husband, she concludes that he must be cheating.) Both wives bring their husbands to the public square on day two.

- If there are three cheating husbands, each wife sees the other cheated-on wives not bring her husband to the public square on day one, or day two. Therefore, she concludes that her own husband is cheating. (The other wives must know someone else is cheating, and that someone else must be her husband.) All three bring their husbands to the public town on day three.
- Thus, by induction, given k cheating husbands, they will all be brought to town on day k.

Question: what did the chief tell the wives that they did not already know?

- Before the chief's statement, every statement of the form "(every wife knows that)^k there is an unfaithful husband" was true for $k \leq 99$ but not for $k = 100$.
- For example, wife 1 knew that 2 knew that 3 knew that ... that 99 knew that 100's husband was cheating, but wife 1 did not know that wife 2 knew that 3 knew that ... that 100 knew that 1's husband was cheating.
- After the meeting, the chief's statement made it common knowledge that there was an unfaithful husband.

Games with incomplete information

A game with incomplete information is a game in which, at the first point in time when the players can begin to plan their moves in the game, some players already have private information about the game what other players do not know.

- It is unnatural to define the beginning of the game to be some point in the distance past before the players learned their private information.
- Furthermore, some parts of a player's private information may be basic to his identity so that it is not even meaningful to talk about him planning his actions before learning this information.

The initial private information that a player has is called the type of the player.

How to represent games with incomplete information (Bayesian games)?

Harsanyi's approach to modelling a Bayesian game, a generalization of the strategic form, in such a way allows game of incomplete information to become games of imperfect information (in which the history of the game is not available to all players).

Representation

Formally, a Bayesian game has the following form

$$\Gamma^b = (N, (C_i)_{i \in N}, (T_i)_{i \in N}, (p_i)_{i \in N}, (u_i)_{i \in N}).$$

- N: a set of players, for each player $i \in N$
- C_i: a set of possible actions,
- T_i: a set of possible types,
- p_i: a probability function, and
- u_i: a utility function.

Γ^b is called finite iff the sets N, C_i and T_i are all finite.

Common knowledge: each player knows the structure of the game and his own actual type in T_i.

Representation

Let $C = \times_{i \in N} C_i$ and $T = \times_{i \in N} T_i$.

- C is the set of all possible profiles or combinations of actions that the players may use in the game, and
- T is the set of all possible profiles or combinations of types that the players may have in the game.

For each player $i \in N$,

$$T_{-i} = \times_{j \in N-j} T_j$$

denotes the set of all possible combinations of types of the players other than i;

$$C_{-i} = \times_{j \in N-j} C_j$$

denotes the set of all possible combinations of actions of the players other than i.

$p_i : T_i \rightarrow \Delta(T_{-i})$, that is, for any possible type t_i in T_i, the probability function must specify a probability distribution $p_i(. \mid t_i)$ over the set T_{-i}, representing what player i could believe about the other players' types if his own type were t_i.

For any $t_{-i} \in T_{-i}$, $p_i(t_{-i} \mid t_i)$ denotes the subjective probability that i would assign to the event that t_{-i} is the actual profile of types of the other players, if his own type were t_i.

Representation

$u_i : C \times T \to R$, that is for any profile of actions and types (c, t) in $C \times T$, the function u_i must specify a number $u_i(c, t)$ that represents the payoff that player i would get, if the players' actual types were all as in t, and the players all choose their actions as specified in c.

Games in Bayesian form:

$$\Gamma^b = (N, (C_i)_{i\in N}, (T_i)_{i\in N}, (p_i)_{i\in N}, (u_i)_{i\in N})$$

Games in strategic form:

$$\Gamma = (N, (C_i)_{i\in N}, (u_i)_{i\in N})$$

Action vs. strategy:

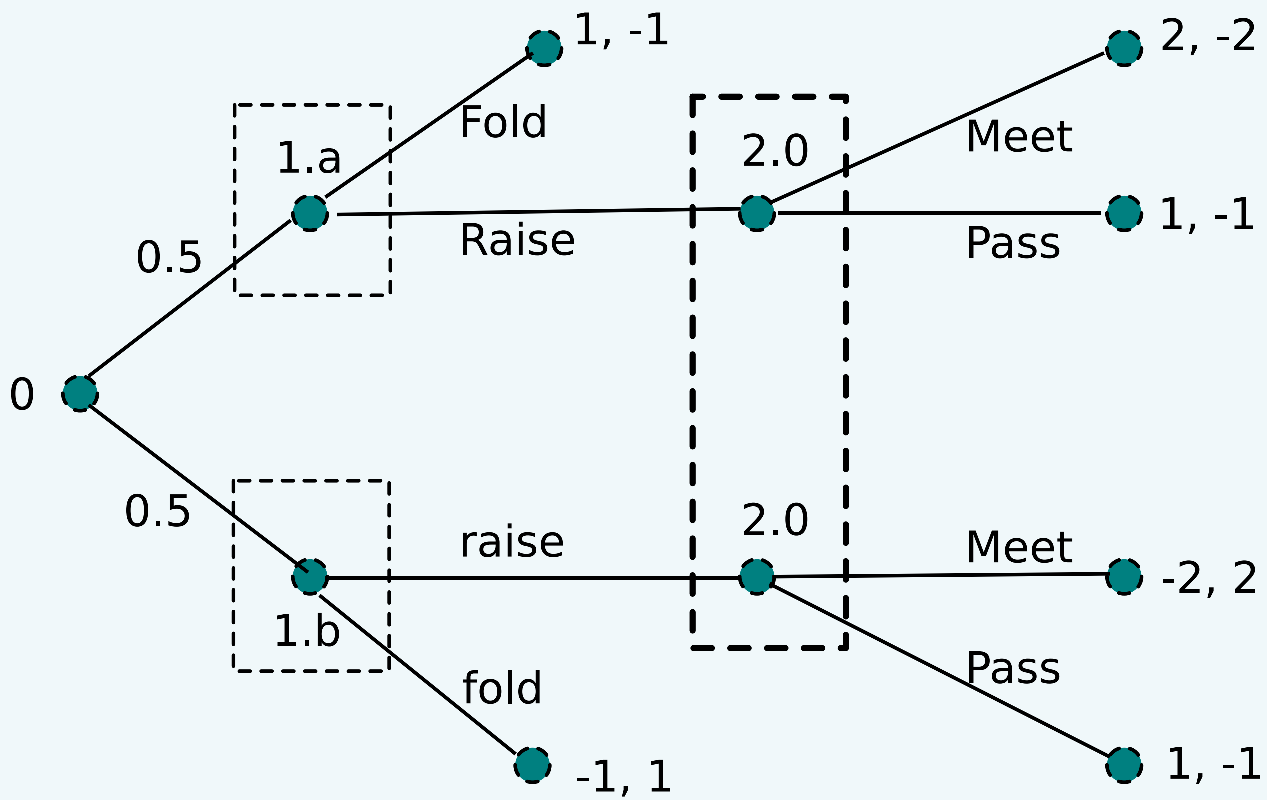
an action in a Bayesian game may represent a plan that specifies a move for every contingency that the player would consider possible after he has learned his type;

a strategy would normally be thought of as a complete plan covering all contingencies that the player would consider possible, before he has learned his type.

A strategy for player i in a Bayesian game is a function from T_i into his set of actions C_i. A strategy must not only specify the actions of the player given the type that he is, but must specify the actions that he would take if he were of another type.

Example

The simple card game is a Bayesian game, if we assume that player 1 already knows the color of the card when the game begins.



Note: this figure is not a game in Bayesian form.

Example

The simple card game is a Bayesian game, if we assume that player 1 already knows the color of the card when the game begins.

- $N = \{1, 2\}$;
- $T_1 = \{1.a, 1.b\}$, $T_2 = \{2.0\}$;
- $C_1 = \{R, F\}$, $C_2 = \{M, P\}$;

- $p_1(2.0 \mid 1.a) = 1.0 = p_1(2.0 \mid 1.b)$;
- $p_2(1.a \mid 2.0) = 0.5 = p_2(1.b \mid 2.0)$;

- $u_1(c, t)$ and $u_2(c, t)$ depend on $(c, t) = (c_1, c_2, t_1, t_2)$.

Example

The simple card game is a Bayesian game, if we assume that player 1 already knows the color of the card when the game begins.

$t_1 = 1.a, t_2 = 2.0$

	M	P
R	2,-2	1,-1
F	1,-1	1,-1

$t_1 = 1.b, t_2 = 2.0$

	M	P
R	-2,2	1,-1
F	-1,1	-1,1

Example

A bargaining game: a seller (player 1) and a buyer (player 2), each knows the object is worth to himself, and thinks that its value to the other maybe an integer between 1 and 100 (euro), each with probability $1/100$. Each player will simultaneously name a bid between 0 and 100. If the buyer's bid is greater than or equal to the seller's bid, then they will trade the object at a price equal to the average of their bids; otherwise no trade will occur.

- $N = \{1, 2\}$;
- $T_1 = T_2 = \{1, 2, \cdots, 100\}$;
- $C_1 = C_2 = \{0, 1, 2, \cdots, 100\}$;
- $p_i(t_{-i} \mid t_i) = 1/100, \ \forall i \in N, \ \forall t = (t_{-i}, t_i) \in T$;
- $u_1(c, t) = (c_1 + c_2)/2 - t_1$ if $c_2 \geq c_1$;
- $u_2(c, t) = t_2 - (c_1 + c_2)/2$ if $c_2 \geq c_1$;
- $u_1(c, t) = u_2(c, t) = 0$ if $c_2 < c_1$.

Consistency in belief

The beliefs $(p_i)_{i \in N}$ in a Bayesian game are consistent iff there exists some probability distribution P in $\Delta(T)$ such that

$$p_i(t_{-i} \mid t_i) = \frac{P(t)}{\sum_{s_{-i} \in T_{-i}} P(s_{-i}, t_i)}, \ \forall t \in T, \ \forall i \in N.$$

For example, in the simple card game, beliefs are consistent with the prior distribution

$$P(1.a, 2.0) = P(1.b, 2.0) = 0.5$$

In the bargaining game,

$$P(t) = 1/10000, \ \forall t \in T = \{1, 2, \cdots, 100\} \times \{1, 2, \cdots, 100\}.$$

All examples have consistent beliefs. Games with inconsistent beliefs do exist!

Equivalence

Two Bayesian games $(N, (C_i)_{i \in N}, (T_i)_{i \in N}, (p_i)_{i \in N}, (u_i)_{i \in N})$ and $(N, (C_i)_{i \in N}, (T_i)_{i \in N}, (q_i)_{i \in N}, (w_i)_{i \in N})$ are fully equivalent iff, for every $i \in N$, there exist function $A_i : T_i \to R$ and $B_i : T \to R$ such that, for every $t_i \in T_i$, $A_i(t_i) > 0$ and

$$q(t_{-i} \mid t_i)w_i(c, t) = A_i(t_i)p(t_{-i} \mid t_i)u_i(c, t) + B_i(t), \ \forall c \in C, \ \forall t_{-i} \in T_{-i}.$$

Every possible type of every player, the two games impute probability and utility functions that are decision-theoretically equivalent. ($A_i(t_i)$ depends on t_i alone, $B_i(t)$ depends on the types of all players.)

Equivalence

Any Bayesian game with finite type set is equivalent to a Bayesian game with consistent beliefs.

Given any Bayesian game $(N, (C_i)_{i \in N}, (T_i)_{i \in N}, (p_i)_{i \in N}, (u_i)_{i \in N})$, we can construct such an equivalent Bayesian game by letting

$$q(t_{-i} \mid t_i) = 1/ \mid T_{-i} \mid \quad \& \quad w_i(c,t) = \mid T_{-i} \mid p_i(t_{-i} \mid t_i) u_i(c,t)$$

for every $i \in N$, $t \in T$, and $c \in C$.

The types are independent and uniformly distributed in the consistent prior of the game $(N, (C_i)_{i \in N}, (T_i)_{i \in N}, (q_i)_{i \in N}, (w_i)_{i \in N})$.

Type-agent representation

$(N, (C_i)_{i \in N}, (T_i)_{i \in N}, (p_i)_{i \in N}, (u_i)_{i \in N}) \Rightarrow (T^*, (D_r)r \in T^*, (v_r)_{r \in T^*})$

One player or agent for every possible type of every player. Assume that $T_i \cap T_j = \emptyset$ if $i \neq j$. The set of player in the type-agent representation is $T^* = \cup_{i \in N} T_i$.

For any $i \in N$ and $t_i \in T_i$, the set of strategies for agent t_i in the type-agent representation is $D_{t_i} = C_i$.

For any $i \in N$ and $t_i \in T_i$, the utility function $v_{t_i} : \text{X}_{s \in T^*} D_s \rightarrow R$ in the type-agent representation is defined so that, for any $d = (d(s))_{s \in T^*}$ in $\text{X}_{s \in T^*} D_s$

$$v_{t_i}(d) = \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i) u_i((d(t_j))_{j \in N}, (t_j)_{j \in N}).$$

(The conditionally expected utility payoff to player i in Γ^b given that t_i is the actual type of player i.)

Difficuties in practical modeling

Practical modelling difficulties arise when players' beliefs are characterized by subjective probabilities, so the question of what one player might believe about another player's subjective probabilities becomes problematic.

Example

“Trivia quiz" game: in the simple card game, the outcome depends on whether player 1 knows the correct answer to some randomly selected question rather than the color of the card.

- Uncertainty: whether player 1 knows the answer to the question
- Player 2’s belief about the uncertainty can be described by Q, his subjective probablity of the event that player 1 knows the answer
- Player 1 may have some uncertainty about player 2’s Q, which can be described by some subjective probablity P
- then player 2 must be able to describe his belief about player 1 belief about Q, which is P, by some subjective probablity Q'
- (goes to infinity) ...

A paradox

The less common knowledge is, the larger the sets of possible types must be, because a player's type is a summary of everything he knows that is not common knowledge.

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To describe a situation in which many individuals have substantial uncertainty about one another's information and beliefs, we may have to develop a complicated Bayesian game model with large type sets and assume that this model is common knowledge among the players.

Is it possible to construct a situation for which there are no sets of types large enough to contain all the private information that players are supposed to have, so that no Bayesian game could represent this situation?

A universal belief space

Under some technical assumption, that no such counterexample to the generality of the Bayesian game model can be constructed, because a universal belief space can be constructed that is always big enough to server as the set of types for each player.

How to construct such universal belief space?

"Readers with less mathematics are encouraged to skim or omit this construction, as nothing later in the book will depend on it."

Questions?

Exercises?!