Game Theory, Chapter 1

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Basic Concepts of Decision Theory

Set of Probability Distributions:

$$\Delta(Z) = \left\{ q : Z \to \mathbb{R} \mid \sum_{y \in Z} q(y) = 1 \text{ and } \forall_{z \in Z} : q(z) \geq 0 \right\}$$

Basic Concepts of Decision Theory

Lottery:

$$\begin{aligned} f : X \times \Omega &\rightarrow \mathbb{R}^+ \\ (x, t) &\mapsto f(x|t) \end{aligned}$$

s.t. for fixed $t \in \Omega$, $f(x|t) \in \Delta(X)$.

$$L = \{f : \Omega \rightarrow \Delta(X)\}.$$

Basic Concepts of Decision Theory

Set of world events: $\Xi = \{S \mid S \subset \Omega, S \neq \emptyset\}$

Preferences in lotteries: $f \gtrsim_S g$, $f \sim_S g$, $f >_S g$

Write: $\alpha f + (1-\alpha)g$ for lottery $\alpha f(x|t) + (1-\alpha)g(x|t)$

Write: $[x]$ for lottery that always gives prize x for sure.

$$\forall t \in \Omega : [x](y|t) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$$

Axioms

A 1 (Completeness/Transitivity)**.**

$$f \gtrsim_S g \;\; or \; g \gtrsim_S f$$

$$f \gtrsim_S g \wedge g \gtrsim_S h \implies f \gtrsim_S h$$

A 2 (Relevance)**.**

$$If \; f(\cdot|t) = g(\cdot|t) \; \forall t \in S, \; then \; f \sim_S g.$$

More Axioms

A 3 (Monotonicity). *If* $f >_S g$, $0 \leq \beta < \alpha \leq 1$, *then*

$$\alpha f + (1 - \alpha) g >_S \beta f + (1 - \beta) g$$

A 4 (Continuity). *If* $f \gtrsim_S g \gtrsim_S h$, *then*

$$\exists_{0 \leq \gamma \leq 1} : g \sim_S \gamma f + (1 - \gamma) h$$

Substitution Axioms

A 5 (Objective Substitution). *If* $e \gtrsim_S f$ *and* $g \gtrsim_S h$, $0 \leq \alpha \leq 1$, *then*

$$\alpha e + (1-\alpha) g \gtrsim_S \alpha f + (1-\alpha) h.$$

A 6 (Subjective Substitution). *If* $f \gtrsim_S g$ *and* $f \gtrsim_T g$, $S \cap T = \emptyset$, *then*

$$f \gtrsim_{S \cup T} g.$$

Strict Substitution Axioms

A 7 (Objective Substitution). *If* $e >_S f$ *and* $g \gtrsim_S h$, $0 < \alpha \leq 1$, *then*

$$\alpha e + (1-\alpha)g >_S \alpha f + (1-\alpha)h.$$

A 8 (Subjective Substitution). *If* $f <_S g$ *and* $f <_T g$, $S \cap T = \emptyset$, *then*

$$f <_{S \cup T} g.$$

Last two.

A 9 (Interest/Regularity). *For every state* $t \in \Omega$ *there exist prizes* $x, y \in X$ *such that* $[x] >_{\{t\}} [y]$.

A 10 (State neutrality). *For all* $r, t \in \Omega$, *if* $f(\cdot|r) = f(\cdot|t)$ *and* $g(\cdot|r) = g(\cdot|t)$ *and* $f \gtrsim_{\{r\}} g$, *then* $f \gtrsim_{\{t\}} g$.

Expected-Utility Maximization

Theorem 1. *All but the last axiom are jointly satisfied if and only if there exists a utility function* $u : X \times \Omega \to \mathbb{R}$ *and a conditional-probability function* $p : \Xi \to \Delta(\Omega)$ *such that*

$$\forall t \in \Omega : \max_{x \in X} u(x,t) = 1 \text{ and } \min_{x \in X} u(x,t) = 0 \quad (1)$$

$$\forall R \subset S \subset T \subset \Omega, S \neq \emptyset : p(R|T) = p(R|S)p(S|T) \quad (2)$$

$$\forall f, g \in L, S \in \Xi : f \gtrsim_S g \Leftrightarrow E_p(u(f)|S) \geq E_p(u(g)|S) \quad (3)$$

In addition, the last axiom is satisfied if and only if (1)-(3) *can be satisfied with a state-independent utility function.*