Equilibria of Strategic-Form Games

Game Theory Seminar

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This week

- 3.1 Domination and Rationalizability
- 3.2 Nash Equilibrium
- 3.3 Computing Nash Equilibria
- 3.5 The Focal Point Effect

Strategic-form game

A strategic-form game Γ is denoted by

$$\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N}).$$

- N: a set of players,
- C_i: a set of possible strategies for player i,
- $u_i : C \to \mathbb{R}$: a utility function for player i,

where C is the set of all possible combinations of strategies:

$$C = \mathsf{X}_{i \in N} C_i.$$

Specify a set of strategies $D_i \subseteq C_i$ for all players that each player might reasonably be expected to use.

Define set of strategies other players might choose:

$$D_{-i} = \times_{j \in N_{-i}} D_j.$$

Let $G_i(D_{-i})$ be the set of all strategies that are such best responses, i.e. $d_i \in G_i(D_{-i})$ iff there exist some $\eta \in \Delta(D_{-i})$ such that

$$d_i \in argmax_{c_i \in C_i} \sum_{d_{-i} \in D_{-i}} \eta(d_{-i}) \cdot u_i(d_{-i}, c_i)$$

A first solution concept

The solution must satisfy

$$D_i \subseteq G_i(D_{-i})$$

Let C_i^∞ denote the strategies for player i after iterative elimination. It can be shown that

$$C_i^\infty = G_i(\times_{j \in N-i} C_j^\infty)$$

Thus, our first and weakest solution concept predicts that the outcome of the game should be a profile of iteratively undominated strategies $\times_{i \in N} C_i^\infty$.

Example 1

	C_2		
C_1	x_2	y_2	z_2
x_1	$3, 0$	$0, 2$	$0, 3$
y_1	$2, 0$	$1, 1$	$2, 0$
z_1	$0, 3$	$0, 2$	$3, 0$

Example 1

C_1	C_2: x_2	y_2	z_2
x_1	$3, 0$	$0, 2$	$0, 3$
y_1	$2, 0$	$1, 1$	$2, 0$
z_1	$0, 3$	$0, 2$	$3, 0$

No strategies are strongly dominated. Therefore $D_1 = \{x_1, y_1, z_1\}$ and $D_2 = \{x_2, y_2, z_2\}$

Notation

Given strategic-form game $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$, we denote:

- Set of pure strategies: C_i.
- Set of randomized strategies: $\Delta(C_i)$.
- Set of randomized strategy profiles: $\text{X}_{i \in N} \Delta(C_i)$.

For each player i, the randomized strategy $\sigma_i \in \Delta(C_i)$ must satisfy:

- $\forall_{c_i \in C_i} \sigma_i(c_i) \geq 0$
- $\sum_{c_i \in C_i} \sigma_i(c_i) = 1$

Expected payoff

For any randomized strategy profile $\sigma \in \mathsf{X}_{i \in N} \Delta(C_i)$ the expected payoff for player i is defined as follows:

$$u_i(\sigma) = \sum_{c \in C} \left(\prod_{j \in N} \sigma_j(c_j) \right) \cdot u_i(c)$$

If player i uses pure strategy d_i, player i's expected payoff is:

$$u_i(\sigma_{-i}, [d_i]) = \sum_{c_{-i} \in C_{-i}} \left(\prod_{j \in N-i} \sigma_j(c_j) \right) \cdot u_i(c_{-i}, d_i)$$

Nash equilibrium

Each player i wants to choose pure strategies that maximize his expected payoff. This means that strategies that do *not* maximize the payoff should have probability 0:

$$\text{if } \sigma_i(c_i) > 0 \text{ then } c_i \in argmax_{d_i \in C_i} u_i(\sigma_{-i}, [d_i]).$$

A randomized strategy profile σ is a Nash equilibrium of Γ if it satisfies this equation for every player i and every strategy $c_i \in C_i$.

Nash equilibrium

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$$\text{if } \sigma_i(c_i) > 0 \text{ then } c_i \in argmax_{d_i \in C_i} u_i(\sigma_{-i}, [d_i]).$$

A randomized strategy profile σ is a Nash equilibrium of Γ if it satisfies this equation for every player i and every strategy $c_i \in C_i$.

Thus, a randomized strategy profile is a Nash equilibrium iff no player could increase his expected payoff by unilaterally deviating from the prediction of the randomized-strategy profile.

Example 1

	C_2		
C_1	x_2	y_2	z_2
x_1	$3, 0$	$0, 2$	$0, 3$
y_1	$2, 0$	$1, 1$	$2, 0$
z_1	$0, 3$	$0, 2$	$3, 0$

Example 1

	C_2		
C_1	x_2	y_2	z_2
x_1	$3, 0$	$0, 2$	$0, 3$
y_1	$2, 0$	$1, 1$	$2, 0$
z_1	$0, 3$	$0, 2$	$3, 0$

- Player 1 might choose x_1, because he expects player 2 to choose x_2.
- Player 2 might choose x_2, because he expects player 1 to choose z_1.

Example 1

C_1	C_2		
	x_2	y_2	z_2
x_1	$3, 0$	$0, 2$	$0, 3$
y_1	$2, 0$	$1, 1$	$2, 0$
z_1	$0, 3$	$0, 2$	$3, 0$

- Player 1 might choose y_1, because he expects player 2 to choose y_2.
- Player 2 might choose y_2, because he expects player 1 to choose y_1.

Example 1

	C_2		
C_1	x_2	y_2	z_2
x_1	$3, 0$	$0, 2$	$0, 3$
y_1	$2, 0$	$1, 1$	$2, 0$
z_1	$0, 3$	$0, 2$	$3, 0$

- Player 1 might choose y_1, because he expects player 2 to choose y_2.
- Player 2 might choose y_2, because he expects player 1 to choose y_1.

In fact, the randomized strategy profile $([y_1], [y_2])$ is an equilibrium of the game.

Example 2

	C_2	
C_1	M	P
Rr	$0, 0$	$1, -1$
Rf	$0.5, -0.5$	$0, 0$
Fr	$-0.5, 0.5$	$1, -1$
Ff	$0, 0$	$0, 0$

Example 2

	C_2	
C_1	M	P
Rr	$0, 0$	$1, -1$
Rf	$0.5, -0.5$	$0, 0$
Fr	$-0.5, 0.5$	$1, -1$
Ff	$0, 0$	$0, 0$

- No equilibria in pure strategies.
- We can expect to find an equilibrium that involves randomization between Rr and Rf and between M and P.
- Let q[Rr] + $(1 - q)$[Rf] and s[M]+$(1 - s)$[P] denote the equilibrium strategies for player 1 and 2.

Example 2

	C_2	
C_1	M	P
Rr	$0, 0$	$1, -1$
Rf	$0.5, -0.5$	$0, 0$
Fr	$-0.5, 0.5$	$1, -1$
Ff	$0, 0$	$0, 0$

Player 1 would be willing to randomize between Rr and Rf only if they give him the same expected payoff against s[M] + $(1 - s)$[P], so

$$0s + 1(1 - s) = 0.5s + 0(1 - s)$$

implying $s = \frac{2}{3}$.

Example 2

	C_2	
C_1	M	P
Rr	$0, 0$	$1, -1$
Rf	$0.5, -0.5$	$0, 0$
Fr	$-0.5, 0.5$	$1, -1$
Ff	$0, 0$	$0, 0$

Player 2 would be willing to randomize between M and P only if they give him the same expected payoff against q[Rr]+$(1-q)$[Rf],so

$$0q + -0.5(1-q) = -1q + 0(1-q)$$

implying $q = \frac{1}{3}$.

Example 2

	C_2	
C_1	M	P
Rr	$0, 0$	$1, -1$
Rf	$0.5, -0.5$	$0, 0$
Fr	$-0.5, 0.5$	$1, -1$
Ff	$0, 0$	$0, 0$

- Implementing the values for q and s gives us the equilibrium ($\frac{1}{3}$[Rr] + $\frac{2}{3}$[Rf], $\frac{2}{3}$[M] + $\frac{1}{3}$[P])
- The expected payoffs are $\frac{1}{3}$ for player 1 and $-\frac{1}{3}$ for player 2.

Example 3

Two suspects are arrested by the police. The police have insufficient evidence for a conviction, and, having separated both prisoners, visit each of them to offer the same deal:

- If one testifies for the prosecution against the other and the other remains silent, the betrayer goes free and the silent accomplice receives the full 6-year sentence.
- If both remain silent, both prisoners are sentenced to only 1 year in jail for a minor charge.
- If each betrays the other, each receives a 5-year sentence.

Each prisoner must make the choice of whether to betray the other or to remain silent. Each one is assured that the other would not know about the betrayal before the end of the investigation. How should the prisoners act?

Example 3

	C_2	
C_1	s_2	b_2
s_1	$5, 5$	$0, 6$
b_1	$6, 0$	$1, 1$

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	C_2	
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The unique Nash equilibrium in this game is ([b_1],[b_2]).

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b_1	$6, 0$	$1, 1$

The unique Nash equilibrium in this game is ([b_1],[b_2]).

Observation: equilibria may be inefficient

Example 4

Players 1 and 2 are husband and wife and have to decide where to go on Saturday afternoon: to the football match or to the shopping center. Neither spouse would derive any pleasure from being without the other, but the husband would prefer to go to the football match whereas the wife would prefer to go to the shopping center.

Example 4

	C_2	
C_1	f_2	s_2
f_1	$3, 1$	$0, 0$
s_1	$0, 0$	$1, 3$

Example 4

	C_2	
C_1	f_2	s_2
f_1	$3, 1$	$0, 0$
s_1	$0, 0$	$1, 3$

There are three equilibria in this game:

- $([f_1], [f_2])$ with expected payoff (3,1).
- $([s_1], [s_2])$ with expected payoff (1,3).
- $(.75[f_1] + .25[s_1], .25[f_2] + .75[s_2])$ with expected payoff $(\frac{3}{4}, \frac{3}{4})$

Example 4

C_1	C_2: f_2	C_2: s_2
f_1	$3, 1$	$0, 0$
s_1	$0, 0$	$1, 3$

There are three equilibria in this game:

- $([f_1], [f_2])$ with expected payoff (3,1).
- $([s_1], [s_2])$ with expected payoff (1,3).
- $(.75[f_1] + .25[s_1], .25[f_2] + .75[s_2])$ with expected payoff ($\frac{3}{4}, \frac{3}{4}$)

Observation: a game may have multiple equilibria.

Support

In a Nash equilibrium, if two different pure strategies of player i both have positive probability, then they must both give him the same expected payoff in the equilibrium.

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The support of a randomized strategy profile $\sigma \in \text{X}_{i \in N} \Delta(C_i)$ is the set of all pure strategy profiles with positive probability if the players choose their strategies according to σ:

$$\text{X}_{i \in N}\{c_i \in C_i | \sigma_i(c_i) > 0\}.$$

Example 4

	C_2	
C_1	f_2	s_2
f_1	$3, 1$	$0, 0$
s_1	$0, 0$	$1, 3$

There are three equilibria in this game:

- $([f_1], [f_2])$
- $([s_1], [s_2])$
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Example 4

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f_1	$3, 1$	$0, 0$
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There are three equilibria in this game:

- $([f_1], [f_2])$
- $([s_1], [s_2])$
- $(.75[f_1] + .25[s_1], .25[f_2] + .75[s_2])$

- The support of the first equilibrium is $\{f_1\}$x$\{f_2\}$
- The support of the second equilibrium is $\{s_1\}$x$\{s_2\}$
- The support of the third equilibrium is $\{f_1, s_1\}$x$\{f_2, s_2\}$

Conditions

To compute a Nash we first make a guess about the support of that equilibrium. We then check whether there is indeed an equilibrium with this support.

For every player i, let D_i be our current guess. If there is an equilibrium σ with support $\times_{i \in N} D_i$, then there must exist numbers $(\omega_i)_{i \in N}$ such that the following conditions are met:

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For every player i, let D_i be our current guess. If there is an equilibrium σ with support $\times_{i \in N} D_i$, then there must exist numbers $(\omega_i)_{i \in N}$ such that the following conditions are met:

Each player must get the same payoff, denoted by ω_i from choosing any of his pure strategies with positive probability:

$$\sum_{c_{-i} \in C_{-i}} \left(\prod_{j \in N-i} \sigma_j(c_j) \right) u_i(c_{-i}, d_i) = \omega_i \qquad \forall i \in N, \quad \forall d_i \in D_i$$

Conditions

Every player i's pure strategies outside D_i get zero probability:

$$\sigma_i(e_i) = 0 \qquad \forall i \in N, \quad \forall e_i \in C_i \backslash D_i$$

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$$\sigma_i(e_i) = 0 \qquad \forall i \in N, \quad \forall e_i \in C_i \backslash D_i$$

For every player i, the probabilities assigned to pure strategies in D_i sum to 1:

$$\sum_{c_i \in D_i} \sigma_i(c_i) = 1 \qquad \forall i \in N.$$

Conditions

The preceding conditions give a system of equations that can be solved. However, the solution may still not be an equilibrium. The following two conditions have to be met:

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The assigned probabilities in d_i must be non-negative:

$$\sigma_i(d_i) \geq 0 \qquad \forall i \in N, \quad \forall d_i \in D_i$$

Conditions

The preceding conditions give a system of equations that can be solved. However, the solution may still not be an equilibrium. The following two conditions have to be met:

The assigned probabilities in d_i must be non-negative:

$$\sigma_i(d_i) \geq 0 \qquad \forall i \in N, \quad \forall d_i \in D_i$$

For every player i, an equilibrium must be better than any pure strategy outside of D_i:

$$\omega_i \geq \sum_{c_{-i} \in C_{-i}} \left(\prod_{j \in N-i} \sigma_j(c_j) \right) u_i(c_{-i}, e_i) \qquad \forall i \in N \quad \forall e_i \in C_i \backslash D_i.$$

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

- There is no equilibrium in which player 1 only chooses one strategy.
- There is no equilibrium in which player 2 only chooses one strategy.

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A first randomized guess is the support $\{T, B\}$x$\{L, M, R\}$

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A first randomized guess is the support $\{T, B\}$x$\{L, M, R\}$

$\omega_1 = 7\sigma_2(L) + 2\sigma_2(M) + 3\sigma_2(R) = 2\sigma_2(L) + 7\sigma_2(M) + 4\sigma_2(R)$

$\omega_2 = 2\sigma_1(T) + 7\sigma_1(B) = 7\sigma_1(T) + 2\sigma_1(B) = 6\sigma_1(T) + 5\sigma_1(B)$

$\sigma_1(T) + \sigma_1(B) = 1, \quad \sigma_2(L) + \sigma_2(M) + \sigma_2(R) = 1$

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A first randomized guess is the support $\{T, B\}$x$\{L, M, R\}$

$\omega_1 = 7\sigma_2(L) + 2\sigma_2(M) + 3\sigma_2(R) = 2\sigma_2(L) + 7\sigma_2(M) + 4\sigma_2(R)$

$\omega_2 = 2\sigma_1(T) + 7\sigma_1(B) = 7\sigma_1(T) + 2\sigma_1(B) = 6\sigma_1(T) + 5\sigma_1(B)$

$\sigma_1(T) + \sigma_1(B) = 1, \quad \sigma_2(L) + \sigma_2(M) + \sigma_2(R) = 1$

$2\sigma_1(T) + 7\sigma_1(B) = 7\sigma_1(T) + 2\sigma_1(B)$ implies $\sigma_1(B) = .5$

$7\sigma_1(T) + 2\sigma_1(B) = 6\sigma_1(T) + 5\sigma_1(B)$ implies $\sigma_1(T) = 3\sigma_1(B)$

Hence, there is no equilibrium with support $\{T, B\}$x$\{L, M, R\}$.

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A second randomized guess is the support $\{T, B\}$x$\{M, R\}$

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A second randomized guess is the support $\{T, B\}$x$\{M, R\}$

$\omega_1 = 2\sigma_2(M) + 3\sigma_2(R) = 7\sigma_2(M) + 4\sigma_2(R)$

$\omega_2 = 7\sigma_1(T) + 2\sigma_1(B) = 6\sigma_1(T) + 5\sigma_1(B)$

$\sigma_1(T) + \sigma_1(B) = 1, \quad \sigma_2(M) + \sigma_2(R) = 1$

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A second randomized guess is the support $\{T, B\}$x$\{M, R\}$

$\omega_1 = 2\sigma_2(M) + 3\sigma_2(R) = 7\sigma_2(M) + 4\sigma_2(R)$

$\omega_2 = 7\sigma_1(T) + 2\sigma_1(B) = 6\sigma_1(T) + 5\sigma_1(B)$

$\sigma_1(T) + \sigma_1(B) = 1, \quad \sigma_2(M) + \sigma_2(R) = 1$

The unique solution to this set of equations is

$\sigma_2(M) = -.25 \qquad \sigma_2(R) = 1.25 \qquad \sigma_1(T) = .75 \qquad \sigma_1(B) = .25$

Thus there is no equilibrium with support $\{T, B\}$x$\{M, R\}$

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A third randomized guess is the support $\{T, B\}$x$\{L, M\}$

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A third randomized guess is the support $\{T, B\}$x$\{L, M\}$

$\omega_1 = 7\sigma_2(L) + 2\sigma_2(M) = 2\sigma_2(L) + 7\sigma_2(M)$

$\omega_2 = 2\sigma_1(T) + 7\sigma_1(B) = 7\sigma_1(T) + 2\sigma_1(B)$

$\sigma_1(T) + \sigma_1(B) = 1, \quad \sigma_2(L) + \sigma_2(M) = 1$

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A third randomized guess is the support $\{T, B\}$x$\{L, M\}$

$\omega_1 = 7\sigma_2(L) + 2\sigma_2(M) = 2\sigma_2(L) + 7\sigma_2(M)$

$\omega_2 = 2\sigma_1(T) + 7\sigma_1(B) = 7\sigma_1(T) + 2\sigma_1(B)$

$\sigma_1(T) + \sigma_1(B) = 1, \quad \sigma_2(L) + \sigma_2(M) = 1$

The unique solution to this set of equations is

$\sigma_2(L) = \sigma_2(M) = .5 \qquad \sigma_2(T) = \sigma_1(B) = .5 \qquad \omega_1 = \omega_2 = 4.5$

However, the pure strategy R for player 2 would give expected payoff 5.5.

Example 5

C_1	C_2: L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A third randomized guess is the support $\{T, B\}$x$\{L, M\}$

$\omega_1 = 7\sigma_2(L) + 2\sigma_2(M) = 2\sigma_2(L) + 7\sigma_2(M)$

$\omega_2 = 2\sigma_1(T) + 7\sigma_1(B) = 7\sigma_1(T) + 2\sigma_1(B)$

$\sigma_1(T) + \sigma_1(B) = 1, \quad \sigma_2(L) + \sigma_2(M) = 1$

The unique solution to this set of equations is

$\sigma_2(L) = \sigma_2(M) = .5 \qquad \sigma_2(T) = \sigma_1(B) = .5 \qquad \omega_1 = \omega_2 = 4.5$

However, the pure strategy R for player 2 would give expected payoff 5.5.

Hence, there is no equilibrium with support $\{T, B\}$x$\{L, M\}$.

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A fourth randomized guess is the support $\{T, B\} \times \{L, R\}$

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A fourth randomized guess is the support $\{T, B\}$x$\{L, R\}$

$$\omega_1 = 7\sigma_2(L) + \sigma_2(R) = 2\sigma_2(L) + 4\sigma_2(R)$$

$$\omega_2 = 2\sigma_1(T) + 7\sigma_1(B) = 6\sigma_1(T) + 5\sigma_1(B)$$

$$\sigma_1(T) + \sigma_1(B) = 1, \quad \sigma_2(L) + \sigma_2(R) = 1$$

Example 5

	C_2		
C_1	L	M	R
T	$7, 2$	$2, 7$	$3, 6$
B	$2, 7$	$7, 2$	$4, 5$

A fourth randomized guess is the support $\{T, B\}$x$\{L, R\}$

$\omega_1 = 7\sigma_2(L) + \sigma_2(R) = 2\sigma_2(L) + 4\sigma_2(R)$

$\omega_2 = 2\sigma_1(T) + 7\sigma_1(B) = 6\sigma_1(T) + 5\sigma_1(B)$

$\sigma_1(T) + \sigma_1(B) = 1, \quad \sigma_2(L) + \sigma_2(R) = 1$

The unique solution to these equations is

$\sigma_2(L) = \frac{1}{6} \quad \sigma_2(R) = \frac{5}{6} \quad \sigma_1(T) = \frac{1}{3} \quad \sigma_1(B) = \frac{2}{3} \quad \omega_1 = \frac{8}{3} \quad \omega_2 = \frac{16}{3}$

Example 5

	C_2		
C_1	L	M	R
T	$7,2$	$2,7$	$3,6$
B	$2,7$	$7,2$	$4,5$

A fourth randomized guess is the support $\{T, B\}$x$\{L, R\}$

$\omega_1 = 7\sigma_2(L) + \sigma_2(R) = 2\sigma_2(L) + 4\sigma_2(R)$

$\omega_2 = 2\sigma_1(T) + 7\sigma_1(B) = 6\sigma_1(T) + 5\sigma_1(B)$

$\sigma_1(T) + \sigma_1(B) = 1, \quad \sigma_2(L) + \sigma_2(R) = 1$

The unique solution to these equations is

$\sigma_2(L) = \frac{1}{6} \quad \sigma_2(R) = \frac{5}{6} \quad \sigma_1(T) = \frac{1}{3} \quad \sigma_1(B) = \frac{2}{3} \quad \omega_1 = \frac{8}{3} \quad \omega_2 = \frac{16}{3}$

The expected payoff to player 2 from choosing M would be $\frac{11}{3} \leq \frac{16}{3}$.
Hence, the equilibrium is ($\frac{1}{3}[T], \frac{2}{3}[B], \frac{1}{6}[L] + \frac{5}{6}[R]$).

Existence theorem

Theorem 1. *Given any finite game Γ in strategic form, there exists at least one equilibrium in $\mathsf{X}_{i \in N} \Delta(C_i)$.*

Existence theorem

Theorem 2. *Given any finite game Γ in strategic form, there exists at least one equilibrium in $\text{X}_{i \in N} \Delta(C_i)$.*

The proof is presented in Section 3.12.

Focal equilibria

A focal equilibrium is an equilibrium that has some property that conspicuously distinguishes it from all the other equilibria.

According to the focal-point effect, if there is one focal equilibrium in a game, then we should expect to observe that equilibrium.

Tradition

	C_2	
C_1	f_2	s_2
f_1	$3, 1$	$0, 0$
s_1	$0, 0$	$1, 3$

There are three equilibria in this game:

- $([f_1], [f_2])$
- $([s_1], [s_2])$
- $(.75[f_1] + .25[s_1], .25[f_2] + .75[s_2])$

Focal arbitrator

A focal arbitrator can determine the focal equilibrium in a game by publicly suggesting to the players that they should all implement this equilibrium.

- Supervisor in a job conflict
- Oldest member of a group
- . . .

Utility payoff

The focal equilibrium can be determined by intrinsic properties of the utility payoffs.

Divide the dollars game: $C_1 = C_2 = \{x \in \mathbb{R} | 0 \geq x \geq 100\}$
with payoff function
$u_i(c_1, c_2) = 0$ if $c_1 + c_2 > 100$
$u_i(c_1, c_2) = c_i$ if $c_1 + c_2 \leq 100$

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For any number x between 0 and 100, the pure strategy pair $(x, 100 - x)$ is an equilibrium. There is also an equilibrium in $(100, 100)$ in which both players have payoff 0.

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For any number x between 0 and 100, the pure strategy pair $(x, 100 - x)$ is an equilibrium. There is also an equilibrium in $(100, 100)$ in which both players have payoff 0.

An impartial arbitrator would probably suggest $(50, 50)$, but even without an arbitrator this equilibrium could be focal.

Focal non-equilibria

The focal-point effect cannot lead intelligent rational players to implement a strategy profile that is not an equilibrium.

	C_2	
C_1	x_2	y_2
x_1	$5, 1$	$0, 0$
y_1	$4, 4$	$1, 5$

Focal non-equilibria

The focal-point effect cannot lead intelligent rational players to implement a strategy profile that is not an equilibrium.

	C_2	
C_1	x_2	y_2
x_1	$5, 1$	$0, 0$
y_1	$4, 4$	$1, 5$

The strategy profile (y_1, x_2) cannot be a self-fulfilling prophecy because if player 1 thought that player 2 would choose x_2, player 1 would choose x_1 instead of y_1.

Conclusion

This week, we have seen:

- A first solution concept to games
- The Nash equilibrium
- Computation of Nash equilibria