Logic Beyond Formulas: A Proof System on Graphs

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Abstract
In this paper we present a proof system that operates on graphs instead of formulas. We begin our quest with the well-known correspondence between formulas and cographs, which are undirected graphs that do not have $P_4$ (the four-vertex path) as vertex-induced subgraph; and then we drop that condition and look at arbitrary (undirected) graphs. The consequence is that we lose the tree structure of the formulas corresponding to the cographs. Therefore we cannot use standard proof theoretical methods that depend on that tree structure. In order to overcome this difficulty, we use a modular decomposition of graphs and some techniques from deep inference where inference rules do not rely on the main connective of a formula. For our proof system we show the admissibility of cut and a generalization of the splitting property. Finally, we show that our system is a conservative extension of multiplicative linear logic (MLL) with mix, meaning that if a graph is a cograph and provable in our system, then it is also provable in MLL+mix.

Keywords: Proof theory, cographs, graph modules, prime graphs, cut elimination, deep inference, splitting, analyticity

1 Introduction
The notion of formula is central to all applications of logic and proof theory in computer science, ranging from the formal verification of software, where a formula describes a property that the program should satisfy, to logic programming, where a formula represents a program [27, 31], and functional programming, where a formula represents a type [25]. Proof theoretical methods are also employed in concurrency theory, where a formula can represent a process whose behaviours may be extracted from a proof of the formula [5, 22, 23, 30]. This formulas-as-processes paradigm is not as well-investigated as the formulas-as-properties, formulas-as-programs and formulas-as-types paradigms mentioned before. In our opinion, a reason for this is that the notion of formula reaches its limitations when it comes to describing processes as they are studied in concurrency theory.

For example, BV [17] and pomset logic [37] are proof systems which extend linear logic with a notion of sequential composition and can model series-parallel orders. However, series-parallel orders cannot express some ubiquitous patterns of causal dependencies such as producer-consumer queues [28], which are within the scope of pomsets [36], event structures [33], and Petri nets [34]. The essence of this problem is already visible when we consider symmetric dependencies, such as separation, which happens to be the dual concept to concurrency in the formulas-as-processes paradigm.

Let us use some simple examples to explain the problem. Suppose we are in a situation where two processes $A$ and $B$ can communicate with each other, written as $A \leq B$, or can be separated from each other, written as $A \not\leq B$, such that no communication is possible. Now assume we have four atomic processes $a$, $b$, $c$, and $d$, from which we form the two processes $P = (a \otimes b) \not\leq (c \otimes d)$ and $Q = (a \not\leq c) \otimes (b \not\leq d)$. Both are perfectly fine formulas of multiplicative linear logic (MLL) [15]. In $P$, we have that $a$ is separated from $b$ but can communicate with $c$ and $d$. Similarly, $d$ can communicate with $a$ and $b$ but is separated from $c$, and so on. On the other hand, in $Q$, $a$ can only communicate with $c$ and is separated from the other two, and $d$ can only communicate with $b$, and is separated from the other two. We can visualize this situation via graphs where $a$, $b$, $c$, and $d$ are the vertices, and we draw an edge between two vertices if they are separated, and no edge if they can communicate. Then $P$ and $Q$ correspond
to the two graphs shown below.
\[
P = (a \otimes b) \boxdot (c \otimes d) \quad Q = (a \boxdot c) \otimes (b \boxdot d)
\]

It should also be possible to describe a situation where \( a \) is separated from \( b \), and \( b \) is separated from \( c \), and \( c \) is separated from \( d \), but \( a \) can communicate with \( c \) and \( d \), and \( b \) can communicate with \( d \), as indicated by the graph below.
\[
\begin{array}{c}
\begin{array}{c}
\quad d \\
\quad c
\end{array} \\
\begin{array}{c}
\quad a \\
\quad b
\end{array}
\end{array}
\]

However, this graph cannot be described by a formula in such a way that was possible for the two graphs in (1). Consequently, the tools of proof theory, that have been developed over the course of the last century, and that were very successful for the formulas-as-properties, formulas-as-processes, and formulas-as-types paradigms, can be used for the formulas-as-processes paradigm only if situations as in (2) above are forbidden. This seems to be a very strong and unnatural restriction, and the purpose of this paper is to propose a way to change this unsatisfactory situation.

We will present a proof system, called GS (for graphical proof system), whose objects of reason are not formulas but graphs, giving the example in (2) the same status as the examples in (1). In a less informal way, one could say that standard proof systems work on cographs (which are the class of graphs that correspond to formulas as in (1)), and our proof systems work on arbitrary graphs. In order for this to make sense, this proof system should obey the following basic properties:

1. **Consistency**: There are graphs that are not provable.
2. **Transitivity**: The proof system should come with an implication that is transitive, i.e., if we can prove that \( A \) implies \( B \) and that \( B \) implies \( C \), then we should also be able to prove that \( A \) implies \( C \).
3. **Analyticity**: As we no longer have formulas, we cannot ask that every formula that occurs in a proof is a subformula of its conclusion. But we can ask that in a proof search situation, there is always only a finite number of ways to apply an inference rule.
4. ** Conservativity**: There should be a well-known logic \( L \) based on formulas such that when we restrict our proof system to graphs corresponding to formulas, then we prove exactly the theorems of \( L \).
5. **Minimality**: We want to make as few assumptions as possible, so that the theory we develop is as general as possible.

Properties 1-3 are standard for any proof system, and they are usually proved using cut elimination. In that respect our paper is no different. We introduce a notion of cut and show its admissibility for GS. Then Properties 1-3 are immediate consequences, and also Property 4 will follow from cut admissibility, where in our case the logic \( L \) is multiplicative linear logic (MLL) with mix [2, 14, 15].

Finally, Property 5 is of a more subjective nature. In our case, we only make the following two basic assumptions:

1. For any graph \( A \), we should be able to prove that \( A \) implies \( A \).
2. If a graph \( A \) is provable, then the graph \( G = C[A] \) is also provable\(^1\), provided that \( C[-] \) is a provable context. This can be compared to the necessitation rule of modal logic, which says that if \( A \) is provable then so is \( \Box A \), except that in our case the \( \Box \) is replaced by the provable graph context \( C[-] \).

All other properties of the system GS follow from the need to obtain admissibility of cut. This means that this paper does not present some random system, but follows the underlying principles of proof theory.

In Section 2, we give preliminaries on cographs, which form the class of graphs that correspond to formulas as in (1). Then, in Section 3 we give some preliminaries on modules and prime graphs, which are needed for our move away from cographs, so that in Section 4, we can present our proof system, which uses the notation of open deduction [18] and follows the principles of deep inference [4, 17, 19]. In Section 5 we show some properties of our system, and Sections 6, 7, and 8 are dedicated to cut elimination. Finally, in Section 9, we show that our system is a conservative extension of MLL+mix.

The contributions of this paper can thus be summarized as follows:

- We present (to our knowledge) the first proof system that is not tied to formulas/cographs but handles arbitrary (undirected) graphs instead.
- We prove a Splitting Lemma (in Section 6), which is often a crucial ingredient in a proof of cut elimination in a deep inference system. But in our case the statement and the proof of this lemma is different from standard deep inference systems, in particular, the general method proposed by Arrer Tubella in her PhD [44] does not apply. But we still use the name Splitting Lemma, as it serves the same purpose.
- We propose a cut rule which corresponds to the standard cut rule in a deep inference system, and show its admissibility. But again, due to the different nature of our proof system, the standard methods must be adapted.

### 2 From Formulas to Graphs

**Definition 2.1.** A *(simple, undirected)* graph \( G \) is a pair \((V_G, E_G)\) where \( V_G \) is a set of vertices and \( E_G \) is a set of

\(^{1}\)Formally, the notation \( G = C[A] \) means that \( A \) is a module of \( G \), and \( C[-] \) is the graph obtained from \( G \) by removing all vertices belonging to \( A \). We give the formal definition in Section 3.
two-element subsets of \( V_G \). We omit the index \( G \) when it is clear from the context. For \( u, w \in V_G \) we write \( uw \) as an abbreviation for \{ \( u, w \) \}. A graph \( G \) is \textit{finite} if its vertex set \( V_G \) is finite. Let \( L \) be a set and \( G \) be a graph. We say that \( G \) is \textit{\( L \)-labelled} (or just \textit{labelled} if \( L \) is clear from context) if every vertex in \( V_G \) is associated with an element of \( L \), called its \textit{label}. We write \( \ell_G(v) \) to denote the label of the vertex \( v \) in \( G \). A graph \( G' \) is a \textit{subgraph} of a graph \( G \), denoted as \( G' \subseteq G \) iff \( V_{G'} \subseteq V_G \) and \( E_{G'} \subseteq E_G \). We say that \( G' \) is an \textit{induced subgraph} of \( G \) if \( G' \) is a subgraph of \( G \) and for all \( u, w \in V_{G'} \), if \( uw \in E_{G'} \) then \( uw \in E_G \). The \textit{size} of a graph \( G \), denoted by \( |G| \), is the number of its vertices, i.e., \( |G| = |V_G| \).

In the following, we will just say \textit{graph} to mean a finite, undirected, labelled graph, where the labels come from the set \( \mathcal{A} \) of atoms which is the (disjoint) union of a countable set of propositional variables \( \mathcal{V} = \{a, b, c, \ldots \} \) and their duals \( \mathcal{V}^\perp = \{a^\perp, b^\perp, c^\perp, \ldots \} \).

Since we are mainly interested in how vertices are labelled, but not so much in the identity of the underlying vertex, we heavily rely on the notion of graph isomorphism.

**Definition 2.2.** Two graphs \( G \) and \( G' \) are \textit{isomorphic} if there exists a bijection \( f : V_G \to V_{G'} \) such that for all \( v, u \in V_G \) we have \( uv = E_G \) iff \( f(v)f(u) = E_{G'} \) and \( \ell_{G'}(v)f = \ell_G(f(v)) \). We denote this as \( G \cong G' \), or simply as \( G \cong G' \) if \( f \) is clear from context or not relevant.

In the following, we will, in diagrams, forget the identity of the underlying vertices, showing only the label, as in the examples in the introduction.

In the rest of this section we recall the characterization of those graphs that correspond to formulas. For simplicity, we restrict ourselves to only two connectives, and for reasons that will become clear later, we use the \textit{\&} (\textit{par}) and \textit{\otimes} (\textit{tensor}) of linear logic [15]. More precisely, \textit{formulas} are generated by the grammar

\[
\phi, \psi :: = \circ \mid a \mid a^\perp \mid \phi \ \& \ \psi \mid \phi \otimes \psi \tag{3}
\]

where \( \circ \) is the \textit{unit}, and \( a \) can stand for any propositional variable in \( \mathcal{V} \). As usual, we can define the negation of formulas inductively by letting \( a^\perp = a \) for all \( a \in \mathcal{V} \), and by using the De Morgan duality between \( \& \) and \( \otimes \): \( (\phi \ \& \ \psi)^\perp = \phi^\perp \otimes \psi^\perp \) and \( (\phi \otimes \psi)^\perp = \phi^\perp \& \psi^\perp \); the unit is self-dual: \( a^\perp = a \).

On formulas we define the following structural equivalence relation:

\[
\begin{align*}
\phi \ \& \ \psi \equiv & \ (\phi \ \& \ \psi) \ \& \ \xi \\
\phi \otimes \psi \equiv & \ (\phi \ \& \ \psi) \otimes \xi \\
\phi \ \& \ \psi \equiv & \ (\phi \ \& \ \psi) \ \\
\phi \otimes \psi \equiv & \ (\phi \otimes \psi) \ \\
\phi \ \& \ \circ \equiv & \ \circ \\
\phi \otimes \circ \equiv & \ \circ
\end{align*}
\]

In order to translate formulas to graphs, we define the following two operations on graphs:

**Definition 2.3.** Let \( G = (V_G, E_G) \) and \( H = (V_H, E_H) \) be graphs with \( V_G \cap V_H = \varnothing \). We define the \textit{par} and \textit{tensor}

operations between them as follows:

\[
\begin{align*}
G \ \& \ H &= \ (V_G \cup V_H, E_G \cup E_H) \\
G \otimes H &= \ (V_G \cup V_H, E_G \cup E_H \cup \{uv \mid u \in V_G, w \in V_H\})
\end{align*}
\]

For a formula \( \phi \), we can now define its associated graph \( [\phi] \) inductively as follows: \( [\circ] = \varnothing \) the empty graph; \( [a] = a \) a single-vertex graph whose vertex is labelled by \( a \) (by a slight abuse of notation, we denote that graph also by \( a \)); similarly \( [a^\perp] = a^\perp \); finally we define \( [\phi \ \& \ \psi] = [\phi] \ \& \ [\psi] \) and \( [\phi \otimes \psi] = [\phi] \otimes [\psi] \).

**Theorem 2.4.** For any two formulas, \( \phi \equiv \psi \iff [\phi] = [\psi] \). \( \square \)

**Definition 2.5.** A graph is \textit{\( P_4 \)-free} (or \textit{\( N \)-free} or \textit{\( Z \)-free}) iff it does not have an induced subgraph of the shape

\[
\begin{align*}
&\bullet \quad \bullet \\
&\text{\bullet}
\end{align*}
\tag{4}
\]

**Theorem 2.6.** Let \( G \) be a graph. Then there is a formula \( \phi \) with \( [\phi] = G \) iff \( G \) is \( P_4 \)-free.

A proof of this can be found, e.g., in [32] or [17].

The graphs characterized by Theorem 2.6 are called \textit{cographs}, because they are the smallest class of graphs containing all single-vertex graphs and being closed under complement and disjoint union.

Because of Theorem 2.6, one can think of standard proof system as \textit{cograph proof systems}. Since in this paper we want to move from cographs to general graphs, we need to investigate, how much of the tree structure of formulas (which makes cographs so interesting for proof theory [26, 38, 42]) can be recovered for general graphs.

## 3 Modules and Prime Graphs

In this section we take some of the concepts that make working with formulas so convenient and lift them to graphs that are not \( P_4 \)-free.

**Definition 3.1.** Let \( G \) be a graph. A \textit{module} of \( G \) is an induced subgraph \( M = (V_M, E_M) \) of \( G \) such that for all \( v \in V_G \setminus V_M \) and all \( x, y \in M \) we have \( vx \in E_G \) iff \( vy \in E_G \).

Modules are used in this paper since they are for graphs what subformulas are for formulas.

**Notation 3.2.** Let \( G \) be a graph and \( M \) be a module of \( G \). Let \( V_C = V_G \setminus V_M \) and let \( C \) be the graph obtained from \( G \) by removing all vertices in \( M \) (including incident edges). Let \( R \subseteq V_C \) be the set of vertices that are connected to a vertex in \( V_M \) (and hence to all vertices in \( M \)). We denote this situation as \( G = C[M]_R \) and call \( C[M]_R \) (or just \( C \)) the \textit{context} of \( M \) in \( G \). Alternatively, \( C[M]_R \) can be defined as follows. If we write \( C[x]_R \) for a graph in which \( x \) is a distinct vertex and \( R \) is the set of neighbours \( x \), then \( C[M]_R \) is the graph obtained from \( C[x]_R \) by substitution of \( x \) for \( M \).
Lemma 3.3. Let \( G \) be a graph and \( M, N \) be modules of \( G \). Then
1. \( M \cap N \) is a module of \( G \).
2. If \( M \cap N \neq \emptyset \), then \( M \cup N \) is a module of \( G \); and
3. if \( N \nsubseteq M \) then \( M \setminus N \) is a module of \( G \).

Proof: The first statement follows immediately from the definition. For the second one, let \( L = M \cap N \neq \emptyset \), and let \( v \in G \setminus (V_M \cup V_N) \) and \( x, y \in V_M \cup V_N \). If \( x, y \) are both in \( M \) or both in \( N \), then we have immediately \( vx \in E_G \) iff \( vy \in E_G \). So, let \( x \in V_M \) and \( y \in V_N \), and let \( z \in L \). We have \( ux \in E_G \) iff \( uz \in E_G \) iff \( vy \in E_G \). Finally, for the last statement, let \( x, y \in V_M \setminus V_N \) and let \( v \in V_G \setminus (V_M \setminus V_N) \). If \( v \notin V_M \), we immediately have \( vx \in E_G \) iff \( vy \in E_G \). So, let \( v \in V_M \), and therefore \( v \in V_M \cap V_N \). Then \( ux \in E_G \) iff \( vx \in E_G \) iff \( vy \in E_G \). \( \square \)

Definition 3.4. Let \( G \) be a graph. A module \( M \) in \( G \) is maximal if for all modules \( M' \) of \( G \) such that \( M \neq M' \), we have \( M \subseteq M' \) implies \( M = M' \).

Definition 3.5. A module \( M \) of a graph \( G \) is trivial iff either \( V_M = \emptyset \) or \( V_M \) is a singleton or \( V_M = V_G \). A graph \( G \) is prime iff \( |V_G| \geq 2 \) and all modules of \( G \) are trivial.

Definition 3.6. Let \( G \) be a graph with \( n \) vertices \( V_G = \{v_1, \ldots, v_n\} \) and let \( H_1, \ldots, H_n \) be \( n \) graphs. We define the composition of \( H_1, \ldots, H_n \) via \( G \), denoted as \( G[H_1, \ldots, H_n] \), by replacing each vertex \( v_i \) of \( G \) by the graph \( H_i \), and there is an edge between two vertices \( x \) and \( y \) if either \( x \) and \( y \) are in the same \( H_i \) and \( xy \in E_{H_i} \) or \( x \in V_{H_i} \) and \( y \in V_{H_j} \) for \( i \neq j \) and \( v_i v_j \in E_G \). Formally, \( G[H_1, \ldots, H_n] = \langle V^*, E^* \rangle \) with

\[
\begin{align*}
V^* &= \bigcup_{1 \leq i \leq n} V_{H_i} \\
E^* &= \bigcup_{1 \leq i \leq n} E_{H_i} \cup \{xy \mid x \in V_{H_i}, y \in V_{H_j}, v_i v_j \in E_G\}
\end{align*}
\]

This concept allows us to decompose graphs into prime graphs (via Lemma 3.7 below) and recover a tree structure for an arbitrary graph, seeing prime graphs as generalized non-decomposable \( n \)-ary connectives. The two operations \( \otimes \) and \( \ominus \), defined in Definition 2.3 are then represented by the two prime graphs.

\[
\begin{align*}
\otimes: & \quad \bullet \quad \bullet \quad \text{and} \quad \ominus: & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{align*}
\]

(5)

If we name these graphs \( \otimes \) and \( \ominus \), respectively, then we can write \( \otimes[G, H] = G \otimes H \) and \( \ominus[G, H] = G \ominus H \).

Lemma 3.7. Let \( G \) be a nonempty graph. Then we have exactly one of the following four cases:
(i) \( G \) is a singleton graph.
(ii) \( G = A \otimes B \) for some \( A, B \) with \( A \neq \emptyset \neq B \).
(iii) \( G = A \ominus B \) for some \( A, B \) with \( A \neq \emptyset \neq B \).
(iv) \( G = P(A_1, \ldots, A_n) \) for some prime graph \( P \) with \( n = |V_P| \geq 4 \) and \( A_i \neq \emptyset \) for all \( 0 \leq i \leq n \).

Proof. Let \( G \) be given. If \( |G| = 1 \), we are in case (i). Now assume \( |G| > 1 \), and let \( M_1, \ldots, M_n \) be the maximal modules of \( G \). Now we have two cases:
- For all \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \) we have \( M_i \cap M_j = \emptyset \). Since every vertex of \( G \) forms a module, every vertex must be part of a maximal module. Hence \( V_G = V_{M_1} \cup \cdots \cup V_{M_n} \). Therefore there is a graph \( P \) such that \( G = P[M_1, \ldots, M_n] \). Since all \( M_i \) are maximal in \( G \), we can conclude that \( P \) is prime. If \( |V_P| \geq 4 \) we are in case (iv). If \( |V_P| < 4 \) we are either in case (ii) or (iii), as the two graphs in (5) are only prime graphs with \( |V_P| = 2 \), and therefore \( G = N \otimes L \otimes K \) or \( G = N \otimes L \otimes L \otimes K \).

4 The Proof System

To define a proof system, we need a notion of implication. To do so, we first introduce a notion of negation.

Definition 4.1. For a graph \( G = \langle V_G, E_G \rangle \), we define its dual \( G^+ = \langle V_G, E_G^+ \rangle \) to have the same set of vertices, and an edge \( uv \in E_G \) iff \( uv \notin E_G \) (and \( v \neq u \)). The label of a vertex \( v \) in \( G^+ \) is the dual of the label of that vertex in \( G \), i.e., \( \ell_{G^+}(v) = \ell_G(v)^c \). For any two graphs \( G \) and \( H \), the implication \( G \rightarrow H \) is defined to be the graph \( G^+ \otimes H \).

Example 4.2. To give an example, consider the graph \( G \) on the left below

\[
G = \begin{array}{c}
a \\
b \quad c
\end{array}
\]

\[
G^+ = \begin{array}{c}
a^+ \\
b^+ \quad c^+
\end{array}
\]

(6)

Its negation \( G^+ \) is shown on the right above.

Observe that the dual graph construction defines the standard De Morgan dualities relating conjunction and disjunction, i.e., for every formula \( \phi \), we have \( [\phi^+] = [\phi]^c \). Furthermore, the De Morgan dualities extend to prime graphs, say \( P \), as \( P[M_1, \ldots, M_n]^+ = P^c[M_1^+, \ldots, M_n^+] \), where \( P^c \) is the dual graph to \( P \). Furthermore, \( P^+ \) is prime if and only if \( P \) is prime. Thus each pair of prime graphs \( P \) and \( P^c \) defines a pair of connectives that are De Morgan duals to each other.

We will now develop our proof system based on the above notion of negation as graph duality. From the requirements mentioned in the introduction it follows that:
(i) for any \( G \), the graph \( G \rightarrow G \) should be provable;
(ii) if \( G \neq \emptyset \) then \( G \) and \( G^p \) should not be both provable;
(iii) the implication \( \rightarrow \) should be transitive, i.e., if \( G \rightarrow H \), and \( H \rightarrow K \) are provable then so should be \( G \rightarrow K \);
(iv) the implication \( \rightarrow \) should be closed under context, i.e., if \( G \rightarrow H \) is provable and \( C[H]^p \) is an arbitrary context, then \( C[G]^p \rightarrow C[H]^p \) should be provable;
(v) if \( A \) and \( C \) are provable graphs, and \( R \subseteq V_C \), then the graph \( C[A]^p \) should also be provable.
Example 4.3. As an example, consider the following three graphs:

\[
\begin{align*}
A_1: & \quad a^+ & b & b^+ \\
A_2: & \quad a^+ & b & b^+ & a \\
A_3: & \quad b & b^+ & a \\
\end{align*}
\]  
(7)

The graph \(A_1\) on the left should clearly be provable, as it corresponds to the formula \((a^+ \supset \exists \ a) \otimes (b \supset \exists \ b^+)\), which is provable in MLL. The graph \(A_3\) on the right should not be provable, as it corresponds to the formula \((a^+ \otimes b) \supset (a \otimes b^+)\), which is not provable in MLL. But what about the graph \(A_2\) in the middle? It does not correspond to a formula, and therefore we cannot resort to MLL. Nonetheless, we can make the following observations. If \(A_3\) were provable, then so would be the graph \(A_4\) shown below:

\[
\begin{align*}
A_4: & \quad a^+ & a \\
\end{align*}
\]  
(8)

as it is obtained from \(A_2\) by a simple substitution. However, \(A_4^+ = A_4\), and therefore \(A_4^+\) and \(A_4\) would both be provable, which would be a contradiction and should be ruled out. Hence, \(A_3\) should not be provable.

We can make further observations without having presented the proof system yet: Notice that \(A_3 \supset A_2\) cannot hold, as otherwise we would be able to use \(A_3\) and modus ponens to establish that \(A_2\) is provable, which cannot hold as we just observed. By applying a dual argument, \(A_2 \supset A_3\) cannot hold. Hence, implication is not simply subset inclusion of edges.\(^2\)

For presenting the inference system we use a deep inference formalism [17, 19], which allows rewriting inside an arbitrary context and admits a rather flexible composition of derivations. In our presentation we will follow the notation of open deduction, introduced in [18].

Let us start with the following two inference rules

\[
\begin{align*}
& \quad \emptyset \quad \emptyset \quad B \otimes A \quad \supset \supset \quad B[A]_S \quad S \subseteq V_B, \ S \ni V_A \\
& \quad i_1 \quad A^+ \quad \supset \supset \quad \exists \supset \quad B[A]_S \quad S \subseteq V_B, \ S \ni V_A \\
& \quad s \supset \supset \quad B \supset A \quad \supset \supset \quad B[A]_S \quad S \subseteq V_B, \ S \ni V_A \\
\end{align*}
\]  
(9)

which are induced by the two Points (i) and (v) above, and which are called identity down and super switch up, respectively. The \(i_1\) says that for arbitrary graphs \(C\) and \(A\) and any \(R \subseteq V_C\), if \(C\) is provable, then so is the graph \(C[A^+ \supset \exists \supset B]\). Similarly, the rule \(s\) says that whenever \(C[B \otimes A]_R\) is provable, then so is \(C[B[A]_S]_R\) for any three graphs \(A, B, C\) and any \(R \subseteq V_C\) and \(S \subseteq V_B\). The condition \(S \neq V_B\) is there to avoid a trivial rule instance, as \(B[A]_S = B \otimes A\) if \(S = V_B\).

Definition 4.4. An inference system \(S\) is a set of inference rules. We define the set of derivations in \(S\) inductively below,

and we denote a derivation \(D\) in \(S\) with premise \(G\) and conclusion \(H\), as follows:

\[
\begin{align*}
G & \quad \vdash \quad S & \quad H \\
\end{align*}
\]

1. Every graph \(G\) is a derivation (also denoted by \(G\)) with premise \(G\) and conclusion \(G\).
2. If \(D_1\) is a derivation with premise \(G_1\) and conclusion \(H_1\), and \(D_2\) is a derivation with premise \(G_2\) and conclusion \(H_2\), then \(D_1 \supset D_2\) is a derivation with premise \(G_1 \supset G_2\) and conclusion \(H_1 \supset H_2\), and similarly, \(D_1 \otimes D_2\) is a derivation with premise \(G_1 \otimes G_2\) and conclusion \(H_1 \otimes H_2\), denoted as

\[
\begin{align*}
G_1 & \quad \vdash \quad S & \quad H_1 \\
G_2 & \quad \vdash \quad S & \quad H_2 \\
\end{align*}
\]

respectively.

3. If \(D_1\) is a derivation with premise \(G_1\) and conclusion \(H_1\), and \(D_2\) is a derivation with premise \(G_2\) and conclusion \(H_2\), and

\[
\begin{align*}
H_1 & \quad \supset \quad G_2 \\
\end{align*}
\]

is an instance of an inference rule \(r\), then \(D_2 \circ D_1\) is a derivation with premise \(G_2\) and conclusion \(H_2\), denoted as

\[
\begin{align*}
G_1 & \quad \vdash \quad S & \quad H_1 \\
G_2 & \quad \vdash \quad S & \quad H_2 \\
\end{align*}
\]

or

\[
\begin{align*}
G_1 & \quad \vdash \quad S & \quad H_1 \\
G_2 & \quad \vdash \quad S & \quad H_2 \\
\end{align*}
\]

If \(H_1 \approx_f G_2\) we can compose \(D_1\) and \(D_2\) directly to \(D_2 \circ D_1\), denoted as

\[
\begin{align*}
G_1 & \quad \vdash \quad S & \quad H_1 \\
G_2 & \quad \vdash \quad S & \quad H_2 \\
\end{align*}
\]

or

\[
\begin{align*}
G_1 & \quad \vdash \quad S & \quad H_1 \\
G_2 & \quad \vdash \quad S & \quad H_2 \\
\end{align*}
\]

or

\[
\begin{align*}
G_1 & \quad \vdash \quad S & \quad H_1 \\
G_2 & \quad \vdash \quad S & \quad H_2 \\
\end{align*}
\]

\(\approx_f\) However, the converse holds in our particular case: We will see later that whenever we have \(G \supset H\) and \(V_B = V_B\) then \(E_H \subseteq E_G\). But this observation is not true in general for logics on graphs. For example in the extension of Boolean logic, defined in [8], it does not hold.
If \( f \) is the identity, i.e., \( H_1 = G_2 \), we can write \( \mathcal{D}_2 \circ \mathcal{D}_1 \) as

\[
\begin{array}{ccc}
G_1 & \downarrow & G_1 \\
\mathcal{D}_1 \parallel s & \mathcal{D}_1 \parallel s \\
H_1 & \mathcal{D}_1 \parallel s & \mathcal{D}_1 \parallel s \\
H_2 & G_2 & G_2 \\
\end{array}
\]

\( \downarrow \) or \( \parallel \)

A proof in \( S \) is a derivation in \( S \) whose premise is \( \emptyset \). A graph \( G \) is provable in \( S \) if \( f \) is a proof in \( S \) with conclusion \( G \). We denote this as \( \vdash G \). A graph \( G \) is a fact beyond the scope of formulas.

\[
A \mathcal{D}_2 \circ \mathcal{D}_1 \mathcal{D}_1 \parallel s \mathcal{D}_1 \parallel s \mathcal{D}_1 \parallel s \mathcal{D}_1 \parallel s
\]

\( \downarrow \) or \( \parallel \)

Remark 4.5. If we have a derivation \( \mathcal{D} \) from \( A \) to \( B \), and a context \( G[\cdot] \), then we also have a derivation from \( G[A] \) to \( G[B] \). We can write this derivation as

\[
\begin{array}{ccc}
G[A] & \downarrow & G[A] \\
\mathcal{D}[\cdot] & \mathcal{D}[\cdot] & \mathcal{D}[\cdot] \\
G[B] & \mathcal{D}[\cdot] & \mathcal{D}[\cdot] \\
\end{array}
\]

\( \downarrow \) or \( \parallel \)

Example 4.6. Let us emphasize that the conclusion of a proof in our system is not a formula but a graph. The following derivation is an example of a proof of length 2, using only \( i \downarrow \) and \( ss \uparrow \):

\[
\begin{array}{ccc}
\emptyset & a \downarrow & b \uparrow \\
i & c \downarrow & d \uparrow \\
\end{array}
\]

\( \downarrow \) or \( \uparrow \)

where the \( ss \uparrow \) instance moves the module \( d \) in the context consisting of vertices labelled \( a, b, c \). The derivation in (11) establishes that the following implication is provable:

\[
a \perp b \quad \alpha \quad a \perp b \\
c \perp d \quad \beta \quad c \perp d
\]

which is a fact beyond the scope of formulas.

As in other deep inference systems, we can give for the rules in (9) their duals, or corules. In general, if

\[
\begin{array}{ccc}
G \downarrow & H \\
\end{array}
\]

is an instance of a rule, then

\[
\begin{array}{ccc}
H \downarrow & G \uparrow \\
i \uparrow & i \downarrow
\end{array}
\]

is an instance of the dual rule. The corules of the two rules in (9) are the following:

\[
\begin{array}{ccc}
d \downarrow & A \otimes A \downarrow \\
i \uparrow & B [\cdot] \downarrow & \emptyset \downarrow \\
\end{array}
\]

\( \perp \) or \( \uparrow \)

called identity up (or cut) and super switch down, respectively. We have the side condition \( S \neq \emptyset \) to avoid a triviality, as \( B [\cdot] \downarrow = B \uparrow A \) if \( S = \emptyset \).

Example 4.7. The implication in (12) can also be proven using only \( ss \downarrow \) and \( i \uparrow \) instead of \( ss \uparrow \) and \( i \downarrow \), as the following proof of length 3 shows:

\[
\begin{array}{ccc}
\emptyset & a \downarrow & b \uparrow \\
i & c \downarrow & d \uparrow \\
\end{array}
\]

\( \downarrow \) or \( \uparrow \)

Definition 4.8. Let \( S \) be an inference system. We say that an inference rule \( r \) is derivable in \( S \) iff

\[
\begin{array}{ccc}
\emptyset & G \downarrow & H \\
i & i \downarrow & \mathcal{D}[\cdot] \\
\end{array}
\]

for every instance \( \vdash G \) there is a derivation \( \mathcal{D}[\cdot] \downarrow \).

We say that \( r \) is admissible in \( S \) iff

\[
\begin{array}{ccc}
\emptyset & G \downarrow & H \\
i & i \downarrow & \mathcal{D}[\cdot] \\
\end{array}
\]

for every instance \( \vdash G \) we have that \( \vdash S \) implies \( \vdash H \).

If \( r \in S \) then \( r \) is trivially derivable and admissible in \( S \). Most deep inference systems in the literature (e.g. \([4, 17, 19, 20, 24, 40]\)) contain the switch rule:

\[
\begin{array}{ccc}
\emptyset & A \uparrow B \otimes C \\
i & A \uparrow B \uparrow C \\
\end{array}
\]

\( \perp \) or \( \uparrow \)

On can immediately see that it is its own dual and is a special case of both \( ss \downarrow \) and \( ss \uparrow \). We therefore have the following:

Lemma 4.9. If in an inference system \( S \) one of the rules \( ss \downarrow \) and \( ss \uparrow \) is derivable, then so is \( s \).

Remark 4.10. In a standard deep inference system for formulas we also have the converse of Lemma 4.9, i.e., if \( s \) is derivable, then so are \( ss \uparrow \) and \( ss \downarrow \) (see, e.g., \([41]\)). However, in the case of arbitrary graphs this is no longer true, and the rules \( ss \uparrow \) and \( ss \downarrow \) are strictly more powerful than \( s \).

Lemma 4.11. Let \( S \) be an inference system. If the rules \( i \downarrow \) and \( i \uparrow \) and \( s \) are derivable in \( S \), then for every rule \( r \) that is derivable in \( S \), also its corule \( r \uparrow \) is derivable in \( S \).
We would like to achieve something similar for our proof system, well-designed deep inference system for formulas, the two rules to our system:

Lemma 4.12. If the rules \( i \uparrow \) and \( s \) are admissible for an inference system \( S \), then \( \rightarrow \) is transitive, i.e., if \( r \vdash S G \rightarrow H \) and \( r \vdash S H \rightarrow K \) then \( r \vdash S G \rightarrow K \).

Proof. We can construct the following derivation

\[
\frac{\frac{}{G \vdash \exists A} \quad \frac{}{H \vdash \exists B} \quad \frac{}{S \vdash \exists C}}{G \vdash \exists C} \quad \frac{G \vdash \exists C}{H \vdash \exists B} \quad \frac{H \vdash \exists B}{S \vdash \exists C}
\]

from \( \emptyset \) to \( \exists \emptyset \) in \( S \).

Lemma 4.12 is the reason why \( i \uparrow \) is also called cut. In a well-designed deep inference system for formulas, the two rules \( i \downarrow \) and \( i \uparrow \) can be restricted in a way that they are only applicable to atoms, i.e., replaced by the following two rules that we call \textit{atomic identity down} and \textit{atomic identity up}, respectively:

\[
a \downarrow a \downarrow a \quad \text{and} \quad a \uparrow a \downarrow \emptyset (16)
\]

We would like to achieve something similar for our proof system on graphs. For this it is necessary to be able to decompose prime graphs into atoms, but the two rules \( ss \downarrow \) and \( ss \uparrow \) cannot do this, as they are only able to move around modules in a graph. For this reason, we add the following two rules to our system:

\[
\frac{(a \downarrow b \downarrow b) \emptyset}{a \downarrow b \downarrow b} \quad \frac{a \downarrow b \downarrow b}{\emptyset}
\]

called \textit{prime down}, and

\[
\frac{P(M \downarrow N) \emptyset}{P(M \downarrow N)}
\]

called \textit{prime up}. In both cases, the side condition is that \( P \) needs to be a prime graph and has at least 4 vertices. We also require that for all \( i \in \{1, \ldots, n\} \) at least one of \( M_i \) and \( N_i \) is nonempty in an application of \( p \downarrow \) and \( p \uparrow \). The reason for these conditions is that the rules would become unsound otherwise, but that the rules are derivable in the general case, as we will see in Lemma 5.2 in the next section.

Example 4.13. Below is a derivation of length 5 using the \( p \downarrow \) rule, and proves that a prime graph implies itself.

\[
\frac{\frac{}{a \downarrow a \downarrow a} \quad \frac{}{b \downarrow b \downarrow b} \quad \frac{}{c \downarrow c \downarrow c} \quad \frac{}{d \downarrow d \downarrow d}}{a \downarrow b \downarrow c \downarrow d}
\]

This completes the presentation of our system, which is shown in Figure 1.

Definition 4.14. We define system SGS to be the set \( \{a \downarrow, ss \downarrow, p \downarrow, p \uparrow, ss \uparrow, ai \downarrow\} \) of inference rules shown in Figure 1. The \textit{down-fragment} (resp. \textit{up-fragment}) of SGS consists of the rules \( \{a \downarrow, ss \downarrow, p \downarrow\} \) (resp. \( \{ai \downarrow, ss \uparrow, p \uparrow\} \)) and is denoted by SGS\( \downarrow \) (resp. SGS\( \uparrow \)). The down-fragment SGS\( \downarrow \) is also called system GS.

5 Properties of the System

The first observation about SGS is that the general forms of the identity rules \( i \downarrow \) and \( i \uparrow \) are derivable, as we show in Lemma 5.1 below. Next, we have a similar result for the prime rules, for which also a general form is derivable, i.e., they can be applied to any graph instead of only prime graphs.

Lemma 5.1. The rule \( i \downarrow \) is derivable in SGS\( \downarrow \), and dually, the rule \( i \uparrow \) is derivable in SGS\( \uparrow \).

Proof. We show by induction on \( G \), that \( G \vdash \exists \emptyset \) has a proof in SGS\( \downarrow \), using Lemma 3.7.

(i) If \( G \) is a singleton graph, we can apply \( ai \downarrow \).
(ii) If \( G = A \vdash B \) then \( G \vdash B \downarrow A^\downarrow \), and we can construct

\[
\frac{\frac{}{D_1 \vdash \exists A^\downarrow B}}{A^\downarrow B \downarrow A}
\]

where \( D_1 \) and \( D_2 \) exist by induction hypothesis.

(iii) If \( G = A \downarrow B \), we proceed similarly.
(iv) If $G = P(A_1, \ldots , A_n)$ for prime $P$ and $|V_P| \geq 4$, we get

$$
\frac{\exists_{\top} \quad \top \quad \top}{\exists_{\top} \quad \top \quad \top}
\frac{\exists_{\top} \quad \top \quad \top}{\exists_{\top} \quad \top \quad \top}
\frac{\exists_{\top} \quad \top \quad \top}{\exists_{\top} \quad \top \quad \top}
\frac{\exists_{\top} \quad \top \quad \top}{\exists_{\top} \quad \top \quad \top}
$$

where $D_1, \ldots , D_n$ exist by induction hypothesis. $\square$

**Lemma 5.2.** For any graph $G$ with $|V_G| = n$, and graphs $M_1, N_1, \ldots , M_n, N_n$, we have derivations

$$
(G_1 \wedge N_1) \otimes \cdots \otimes (M_i \wedge N_i) \wedge (G_i \vee N_i)
$$

and dually

$$
(G(1, \ldots , M_n) \otimes G(1, \ldots , N_n)) \wedge (M_1 \otimes N_1) \wedge \cdots \wedge (M_n \otimes N_n)
$$

**Proof:** We only show (19), and proceed by induction on the size of $G$, using Lemma 3.7.

(i) If $G$ is a singleton graph, the statement holds trivially.

(ii) If $G = A \vee B$ then $G(N_1, \ldots , N_n) = A(N_1, \ldots , N_k) \wedge B(N_{k+1}, \ldots , N_n)$ for some $1 \leq k \leq n$. We therefore have

$$
\frac{(M_i \vee N_i) \otimes (M_i \vee N_i)}{D_i \perp S_{\top} \perp S_{\top}}
\frac{(M_i \vee N_i) \otimes (M_i \vee N_i)}{D_i \perp S_{\top} \perp S_{\top}}
\frac{(M_i \vee N_i) \otimes (M_i \vee N_i)}{D_i \perp S_{\top} \perp S_{\top}}
\frac{(M_i \vee N_i) \otimes (M_i \vee N_i)}{D_i \perp S_{\top} \perp S_{\top}}
$$

where $D_1$ and $D_2$ exist by induction hypothesis.

(iii) If $G = A \otimes B$, we proceed similarly.

(iv) If $G = P(A_1, \ldots , A_n)$ for prime $P$ and $|V_P| \geq 4$, we have an instance of $P$.

The derivation in (20) can be constructed dually. $\square$

Next, observe that Lemmas 4.11 and 4.12 hold for system SGS. In particular, we have that if $\vdash_{\top} A \rightarrow B$ and $\vdash_{\top} B \rightarrow C$ then $\vdash_{\top} A \rightarrow C$ because $\vdash_{\top} \in S_{\top}$. The main result of this paper is that Lemma 4.12 does also hold for GS. More precisely, we have the following theorem:

**Theorem 5.3 (Cut Admissibility).** The rule $\vdash_{\top}$ is admissible for GS.

To prove this theorem, we will show that the whole up-fragment of SGS is admissible for GS.

**Theorem 5.4.** The rules $\vdash_{\top}, S_{\top}$, and $P_{\top}$ are admissible for GS.

Then Theorem 5.3 follows immediately from Theorem 5.4 and the second statement in Lemma 5.1.

The following three sections are devoted to the proof of Theorem 5.4. But before, let us finish this section by exhibiting some immediate consequences of Theorem 5.3.

**Corollary 5.5.** For every graph $G$, we have $\vdash_{\top} A$ if $\vdash_{\top} A$.

**Corollary 5.6.** For all graphs $G$ and $H$, we have

$$
\frac{G \vee H}{G \perp H} \quad \frac{G \perp H}{G \vee H} \quad \frac{G \perp H}{G \vee H}
$$

**Proof:** The first equivalence is just the definition of $\vdash_{\top}$. The second equivalence follows from Theorem 5.4, and the last equivalence follows from the two derivations

$$
\frac{G \vee H}{G \perp H} \quad \frac{G \perp H}{G \vee H}
$$

and together with Lemma 5.1. $\square$

**Corollary 5.7.** We have $\vdash_{\top} A \rightarrow B$ and $\vdash_{\top} A$ and $\vdash_{\top} B$.

**Proof:** This follows immediately by inspecting the inference rules of GS. $\square$

**Corollary 5.8.** We have $\vdash_{\top} P(M_1, \ldots , M_n)$ with $P$ prime and $n \geq 4$ and $M_i \neq \emptyset$ for all $i = \{1, \ldots , n\}$, if and only if there is at least one $i = \{1, \ldots , n\}$ such that $\vdash_{\top} M_i$ and $\vdash_{\top} P(M_1, \ldots , M_{i-1}, \emptyset , M_{i+1}, \ldots , M_n)$.

This can be seen as a generalization of the previous corollary, and it is proved similarly.

**Remark 5.9.** The system GS forms a proof system in the sense of Cook and Reckhow [10], as the time complexity of checking the correct application of inference rules is polynomial, since the modular decomposition of graphs can be
obtained in linear time [29]. Also whenever graph isomorphism is used to compose derivations, as in (10), we assume that the isomorphism \( f \) is explicitly given.

**Theorem 5.10.** Provability in GS is decidable and in NP.

*Proof.* This follows immediately from the observation that to each graph only finitely many inference rules can be applied, and that the length of a derivation in GS is \( O(n^3) \) where \( n \) is the number of vertices in the conclusion. This can be seen as follows: every inference rule application in GS, when seen bottom-up, removes either two vertices or at least one edge. No rule can introduce vertices or edges. \( \square \)

### 6 Splitting

The standard syntactic method for proving cut elimination in the sequent calculus is to permute the cut rule upwards in the proof and decomposing the cut formula along its main connective, and so inductively reduce the cut rank. However, in our proof system this method cannot be applied, as derivations can be constructed in a more flexible way than in the sequent calculus. For this reason, the splitting technique has been developed in the literature on deep inference [17, 21, 24, 41]. However, since we are working on general graphs instead of formulas, the generic method developed by Aler Tubella [44], cannot directly be applied in our case. For this reason, we needed to adapt the method and prove all lemmas from scratch. The central lemma is the following:

**Lemma 6.1 (Splitting).** Let \( G, A, B \) be graphs, let \( P \) be a prime graph with \( n = |V_P| \geq 4 \), let \( M_1, \ldots, M_n \) be nonempty graphs, and let \( a \) be a atom.  

1. If \( r_{GS} G \not\not\not\not A \otimes B \) then there are a context \( C[\cdot]_R \) and graphs \( K_A \) and \( K_B \) such that there are derivations  
   \[ C[K_A \otimes K_B] \mid G, D_A \mid GS, D_B \mid GS, D_C \mid GS \mid C, D_K \mid GS \mid K_B \not\not\not\not Y B \mid K \not\not\not\not Y M_i \]

2. If \( r_{GS} G \not\not\not\not P[\emptyset, \ldots, M_n] \), then there are
   - either a context \( C[\cdot]_R \) and graphs \( K_1, \ldots, K_n \), such that there are derivations  
     \[ C[P \{K_1, \ldots, K_n\}] \mid G, D_C \mid GS, D_K \mid GS \mid K \not\not\not\not Y M_i \]
   - or a context \( C[\cdot]_R \) and graphs \( K_X \) and \( K_Y \) such that there are derivations  
     \[ C[K_X \not\not\not\not Y K_Y] \mid G, D_C \mid GS, D_K \mid GS \mid K \not\not\not\not Y M_i \]

Note that in the statement of Lemma 6.1, the first case (1) is superfluous, as it is a special case of (2), when we see \( \otimes \) as a prime graph, as indicated in (5) in Section 3. In this case the two subcases of (2) collapse. We nonetheless decided for pedagogical reasons to list case (1) explicitly. It shows how our splitting lemma is related to the standard splitting lemmas in the deep inference literature [17, 19, 21, 24, 41, 44], and thus enables the reader to see where the two subcases in case (2) come from.

The idea of splitting is that, in a provable “sequent-like graph”, consisting of a number of disjoint connected components, we can select any of these components as the principal component and apply a derivation to the other components, such that eventually a rule breaking down the principal component can be applied. This allows us to approximate the effect of applying rules in the sequent calculus.

We will use the proof in (14) as an example to explain this idea. In the conclusion we have 3 connected components. We can select the \( N \)-shape component on the right as the principal component, and apply case (2) of Lemma 6.1 to reorganise the proof (14) such that an instance of \( p_i \) involving the \( N \)-shape can be applied, as in Example 4.13. The bottom-step of such a reorganised proof is shown below:

If, on the other hand, we pick in (14) the \( d^\perp \) as principal component, and apply case (3) of Lemma 6.1, we get the following derivation

\[
\begin{array}{c}
\begin{array}{c}
d^\perp \not\not\not\not (a \perp \emptyset) \otimes \emptyset \otimes b \perp \emptyset \otimes c \perp \emptyset \otimes d \end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
a \not\not\not\not \emptyset \\
a \perp b \perp c \perp d
\end{array}
\end{array}
\]

which we can complete to a proof with an an application of the rule \( a\perp \).
A significant departure from established splitting lemmas in the literature is the need for contexts in the premises of derivations. This is required to cope with graphs such as the following:

\[
\begin{array}{c}
\bullet & a & b & c \\
\downarrow & b & a & c \\
\end{array}
\]  \hspace{1cm}
\begin{array}{c}
\bullet & a & b & c \\
\downarrow & a & b & c \\
\end{array}
\]  \hspace{1cm}
\begin{array}{c}
\bullet & a & b & c \\
\downarrow & a & b & c \\
\end{array}
\]  \hspace{1cm}
\begin{array}{c}
\bullet & a & b & c \\
\downarrow & a & b & c \\
\end{array}
\]

If we take \(a\) as the principal component, and apply case (3) of Lemma 6.1, we get a nonempty context \(C\). Notice, furthermore, the above graph is provable only by applying rules deep inside the modular decomposition of the graph, as follows:

\[
\begin{array}{c}
\bullet & a & b & c \\
\downarrow & a & b & c \\
\end{array}
\]  \hspace{1cm}
\begin{array}{c}
\bullet & a & b & c \\
\downarrow & a & b & c \\
\end{array}
\]  \hspace{1cm}
\begin{array}{c}
\bullet & a & b & c \\
\downarrow & a & b & c \\
\end{array}
\]  \hspace{1cm}
\begin{array}{c}
\bullet & a & b & c \\
\downarrow & a & b & c \\
\end{array}
\]

This shows that deep inference is necessary for this kind of proof theory on graphs.

The second subcase in case (2) of Lemma 6.1 is required for examples such as the following:

\[
\begin{array}{c}
\bullet & b & a & b \\
\downarrow & b & a & b \\
\end{array}
\]  \hspace{1cm}
\begin{array}{c}
\bullet & b & a & b \\
\downarrow & b & a & b \\
\end{array}
\]  \hspace{1cm}
\begin{array}{c}
\bullet & b & a & b \\
\downarrow & b & a & b \\
\end{array}
\]

If we select the \(N\)-shape as the principal component and try to apply \(a\), then \(a\) and \(a^+\) can no longer communicate. Therefore, we must first move \(b^+\) or \(c^+\) into the structure and apply an \(a\) or \(a^+\), in order to destroy the prime graph. For example, by using \(b^+\) to cancel out \(b\), we obtain a provable graph of the form \(a^+ (c \otimes a^+) \otimes c^+\).

The proof of Lemma 6.1 proceeds by induction on the size of the derivation by exhaustively considering all ways in which the bottommost rule can interact with the principal component. It can be found in Appendix A.

### 7 Context Reduction

The Splitting Lemma 6.1 only applies in a shallow context, i.e., the outermost nodes in the modular tree construction of a graph (see Lemma 3.7). In order to use splitting for cut elimination, we need to apply it in arbitrary contexts. For this we need the context reduction lemma.

**Lemma 7.1 (Context reduction).** Let \(A\) be a graph and \(G[\cdot]_S\) be a context. If \(\tau_{GS} G[A]_S\) then there is a graph \(K\) and a context \(C[\cdot]_R\) such that there are derivations

\[
\begin{array}{c}
\emptyset & D_C \parallel GS \\
\emptyset & D_A \parallel GS \\
C[\cdot]_R & D_K \parallel GS
\end{array}
\]

for any graph \(X\).

The proof of this lemma proceeds by a case analysis on the structure of the context \(G[\cdot]_S\), employing splitting at each step. It can be found in Appendix B. We show here only one case.

Assume \(G[A]_S = G'' \otimes P[M_1[A]_{S'}, M_2, \ldots, M_n]\) for some \(G''\), prime graph \(P\) and \(M_1, \ldots, M_n\). Applying Lemma 6.1(2) gives us three different cases, of which we show here only one: We get \(C[\cdot]_R\) and \(K\) and \(K\), such that

\[
\begin{array}{c}
C'[\cdot]_R \parallel D_C \parallel G'' \\
\emptyset \parallel \emptyset \parallel D_K \parallel K \otimes M_1[A]_{S'} \parallel K \otimes M_2 \parallel \ldots \parallel K \otimes M_n
\end{array}
\]

We apply the induction hypothesis to \(D_K\) and get \(K\) and \(C''[\cdot]_R\), such that

\[
\begin{array}{c}
\emptyset \parallel D_C \parallel G'' \\
\emptyset \parallel D_K \parallel K \otimes M_1[X]_{S'} \parallel K \otimes M_2 \parallel \ldots \parallel K \otimes M_n
\end{array}
\]

for any \(X\). We let \(C[\cdot]_R = C'[\cdot] \otimes P[C''[\cdot]_R, M_2, \ldots, M_n]\) and obtain \(D_C\) via

\[
\begin{array}{c}
\emptyset \parallel D_C \parallel G'' \\
\emptyset \parallel D_K \parallel K \otimes M_1[X]_{S'} \parallel K \otimes M_2 \parallel \ldots \parallel K \otimes M_n
\end{array}
\]

The other cases follow by a similar reasoning.

### 8 Elimination of the Up-Fragment

In this section we discuss how we use splitting and context reduction to prove Theorem 5.4, i.e., the admissibility of the rules \(a\), \(ss\), and \(p\). For the rules \(a\), \(ss\), and \(p\), this is similar to ordinary deep inference systems (see, e.g., [9, 21, 24, 41]). But for \(p\), there are surprising differences. In particular, we need to invoke an induction on the “size of the cut formula”. In other cut elimination proofs in deep inference, there is no
need for such an induction, as it is outsourced to the splitting lemma. Consider an instance of $p \uparrow$, as follows.

$$
\begin{align*}
 p \uparrow & : G(M_1 \land \ldots \land M_n) \rightarrow P^\perp(N_1 \land \ldots \land N_n) \\
& \text{for any graph } X. \text{ We apply Lemma 6.1.(1) to } D_2 \text{ and get graphs } L_P \text{ and } L_{P^\perp} \text{ and a context } C_2 \vdash \Delta_i \text{ such that}
\end{align*}
$$

Here, we define the size of such an instance of $p \uparrow$ as

$$
\sum_{1 \leq i \leq n} (|M_i| + |N_i|)
$$

i.e., the number of vertices in the subgraph that is modified by the rule. To prove admissibility of $p \uparrow$, assume we have a proof of $G(M_1 \land \ldots \land M_n) \rightarrow P^\perp(N_1 \land \ldots \land N_n)$. We apply Lemma 7.1 and get a graph $L$ and a context $C_1 \vdash \Delta_i$, such that there are derivations

$$
\begin{align*}
& \Delta_i \vdash C_1, \\
& L \vdash (P(M_1, \ldots, M_n) \rightarrow P^\perp(N_1, \ldots, N_n)), \\
& C_1 \vdash [L \not\vdash (X)]_{\text{ss}} \vdash
\end{align*}
$$

for any graph $X$. Applying Lemma 6.1.(2) to $D_6$ and $D_7$ gives us four different cases, according to the two possible outcomes of case (2) in Lemma 6.1. We show here only the most complicated one, in which we get $K_Z$ and $K_W$ and $H_X$ and $H_Y$ and contexts

\[\vdash \Delta_i \vdash C_2, \quad \Delta_i \vdash L_P \not\vdash P(M_1, \ldots, M_n) \quad \text{and} \quad \Delta_i \vdash L_{P^\perp} \not\vdash P^\perp(N_1, \ldots, N_n)\].

The complete proof, together with the proofs for ai $\uparrow$ and ss $\uparrow$ can be found in Appendix C.
Lemma 9.2. Let \( A \) and \( B \) be cographs. Then
\[
\text{ss}\hspace{1pt} \frac{A}{B} \quad \Rightarrow \quad \frac{A}{\{s\}}
\]  

Proof. By Theorem 2.6, the graphs \( A \) and \( B \) are cographs iff there are formulas \( \phi \) and \( \psi \) with \( \llbracket \phi \rrbracket = A \) and \( \llbracket \psi \rrbracket = B \). Now the statement follows from the corresponding statement for formulas (see e.g., Lemma 4.3.20 in [41]).

\[\square\]

Theorem 9.3. Let \( A \) be a cograph. Then \( \vdash_{\text{GS}} A \iff \vdash_{\{a\downarrow, s\}} A \).

Proof. The implication from right to left follows immediately from the fact that \( s \) is a special case of \( \text{ss}\downarrow \) (see Lemma 9.1). For the implication from left to right, apply Lemma 9.2 to get a derivation \( D \) that only uses cographs. Hence the rule \( p_1 \) is not used in \( D \). Therefore, by Lemma 9.2, we can get a derivation \( D' \) that uses the rules \( a\downarrow \) and \( s \).

\[\square\]

Corollary 9.4. For any unit-free formula \( \phi \),
\[
\vdash_{\text{MLX}} \phi \iff \vdash_{\text{GS}} \llbracket \phi \rrbracket
\]

Proof. It has been shown before (see, e.g., [19, 41]) that a unit-free formula \( \phi \) is provable in MLLX if it is provable in \( \{a\downarrow, s\} \) (note that in (15) we can have \( B = \emptyset \)). Now the statement follows from Theorem 9.3 and Theorem 2.6.

\[\square\]

Corollary 9.5. Provability in GS is NP-complete.

Proof. Since MLLX is NP-complete, we can conclude from Corollary 9.4 that GS is NP-hard. Containment in NP has been proved in Theorem 5.10.

\[\square\]

10 Discussion and related work

Here we draw attention to challenges surrounding GS. Using examples, such as (22) and (24), we have already explained why GS necessarily demands deep inference. Since no established deep inference system matches GS we have a fundamentally new proof system. Furthermore, we explain in this section that simply taking an established semantics for MLLX based on graphs and dropping the restriction to cographs does not immediately yield a semantics for GS.

Criteria for proof nets. Graphical approaches to proof nets such as R&B-graphs [38] have valid definitions when we drop the restriction to cographs. However, we show that (at least without strengthening criteria), these definition do not yield a semantics for a logic over graphs, since logical principles laid out in the introduction are violated.

Consider again graph (8), which is not provable in GS. In an R&B-graph we draw blue edges representing the axiom links of proof nets, as shown below for graph (8).

\[
\begin{array}{c}
a \\
\downarrow
\end{array}
\begin{array}{c}
a \\
\downarrow
\end{array}
\]

\[\square\]

The established correctness criterion for R&B-graphs would wrongly accept the above graph. The reason is the cycle of 4
nodes alternating between red and blue edges has a chord. Notice this observation is independent of the rules of the system GS, since, in Sec. 4, we showed that graph (8) cannot be provable in a system subject to the logical principle of consistency.

What about cliques and stable sets? The switch rule has the property that it reflects edges and maximal cliques. That is: if there is an edge in the conclusion it will also appear in the premise and every maximal clique in the premise is a superset of some maximal clique in the conclusion. Indeed, mappings reflecting maximal cliques and preserving stable sets (mutually independent nodes) have a long history in program semantics [3] which led to coherence spaces and the discovery of linear logic [15], see also [8, 12, 13]. Therefore it is a reasonable starting point to try generalising switch by using such maximal clique reflecting homomorphisms, instead of ss↓. Indeed this is how we discovered ss↓, which is sound with respect to such homomorphisms.

Unfortunately, replacing ss↓ with maximal clique reflecting homomorphisms yields a system distinct from our graphical system, for example the following would be provable, but is not provable in GS.

\[
\begin{align*}
\text{We may try replacing both } ss↓ & \text{ and } ss↑ \text{ using a stronger symmetric notion of homomorphism where, in addition, every maximal stable set in the conclusion is a superset of some maximal stable set in the premise. Using such a homomorphism which is both maximal clique reflecting and stable set preserving as a rule, the above example is not provable. To see why, observe that at some point either } a \text{ and } \overline{a} \text{ or } b \text{ and } \overline{b} \text{ must be brought together into a module where they can interact, but this cannot be achieved while preserving the maximal stable set } \{a, b, \overline{a}\}. \\
\text{Notice however, that if we replace } ss↓ & \text{ and } ss↑ \text{ by the symmetric homomorphism described above, the implication below would be provable.}
\end{align*}
\]

\[
\begin{align*}
\text{In contrast, the above is not provable in GS, since both sides are distinct prime graphs; and there is no suitable way to apply } ss↓. \text{ Thus, we would obtain a distinct system from GS by using such homomorphisms.}
\end{align*}
\]

Studying logics coming out of reflecting maximal cliques and preserving maximal stable sets is currently a topic of active research and leads to possible extensions of Boolean logic to graphs [7, 8, 45].

Generalised connectives. In this paper, we use a modular decomposition of graphs based on prime graphs (see Lemma 3.7). The connectives \( \forall \) and \( \otimes \) are given by the prime graphs on two vertices. This choice is coherent with the graphs operations of union, join and composition, i.e. \( G \otimes H = \otimes(G, H) \) and \( G \forall H = \forall(G, H) \). Pushing forward this idea, any graph can be interpreted as a (multiplicative) generalized connective \([1, 11, 16]\). In particular, in light of Lemma 3.7, every prime graph defines a non-decomposable connective. Furthermore, our Lemma 3.7 also provides a more refined notion of decomposition than the \( \forall-\otimes \) decomposition known in the literature. However, the exact relation between the two constructions requires further investigation. Note, for example, that the number of pairs of orthogonal 6-ary non-decomposable connectives known at the time of writing is strictly smaller than the number of pairs of dual prime graphs on 6 vertices. Nonetheless, we conjecture that there is a correspondence between connectives defined by means of orthogonal sets of partitions and connectives defined by means of graphs.

11 Conclusion
Guided by logical principles, we have devised a minimal proof system (GS in Fig. 1) that operates directly over graphs, rather than formulas. Negation is generalised in terms of graph duality, while disjunction is disjoint union of graphs, allowing us to define implication “\( G \text{ implies } H \)” as the standard “not \( G \text{ or } H \)” (see Def. 4.1). All other design decisions are then fixed by our guiding logical principles. Most of these principles follow from cut elimination (Theorem 5.3), to which the majority of this paper is dedicated. We also confirm that GS conservatively extends MLL\(_X\) (Corollary 9.4) — a logic at the core of many proof systems.

Surprisingly, even for such a minimal generalisation of logic to graphs, deep inference is necessary. Proof systems for classical logic, MLL\(_X\) and many other logics may be expressed using deep inference, but deep inference is generally not necessary, since many standard logics have presentations in the sequent calculus where all inferences are applied at the root of some formula in a sequent. In contrast, for some logics (e.g., BV \([17, 43]\] and modal logic S5 \([35, 39]\)), deep inference is necessary in order to define a proof system satisfying cut elimination. System GS goes further than the aforementioned systems in that all intermediate lemmas such as splitting (Lemma 6.1) and context reduction (Lemma 7.1) also demand a deep formulation, requiring additional context awareness. As such we were required to generalise the basic mechanisms of deep inference itself in order to establish cut elimination (Theorem 5.3) for a logic over graphs. This is due to a property of general graphs that is forbidden in formulas — that the shortest path between any two connected nodes may be greater than two; and hence, when we apply
reasoning inside a module (i.e., a context), there may exist paths of dependencies that indirectly constrain the module.

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References


A Proof of Splitting (Lemma 6.1)

Observation A.1. Whenever we have a derivation from $A$ to $B$ in GS, then $|A| \leq |B|$, as the rules $ss \downarrow$ and $p \downarrow$ do not change the size of a graph, and the rule $a \downarrow$ deletes two vertices when going up in a derivation.

Lemma A.2. Let $C_i[\cdot]_{R_i}, \ldots, C_n[\cdot]_{R_n}$ be contexts. If $\Gamma_{\text{GS}} C_i$ for all $i \in \{1, \ldots, n\}$, then $\Gamma_{\text{GS}} C_i[\cdots C_2[\cdot]_{R_2}\cdots]_{R_n}$.

Proof. We proceed by induction on $n$. The base case for $n = 1$ is trivial, and the inductive case for $n > 1$ is this derivation:

$$
C_n[\cdot]_{R_n} \quad \vdash D_n \parallel \quad C_{n-1}[\cdot]_{R_2} \quad \vdash D_{n-1} \parallel \quad \cdots
$$

where $D_n$ is the derivation for $C_n$ and $D'$ exists by induction hypothesis. \qed

Proof of Lemma A.2. We prove all three statements simultaneously, by induction on the pair $(|F|, |D|)$, ordered lexicographically, where $F$ is the graph provable in the premise of each statement (i.e., $F = G \mathrel{\mathcal{N}} (A \otimes B)$ in (1), and $F = G \mathrel{\mathcal{N}} P(M_1, \ldots, M_n)$ in (2), and $F = G \mathrel{\mathcal{N}} a$ in (3)) and $D$ is the proof of $F$.

1. Assume we have a proof $D$ of $G \mathrel{\mathcal{N}} (A \otimes B)$. We have to find graphs $K_A$ and $K_B$, and a context $C[\cdot]_{R}$, and derivations $D_G$, $D_C$, $D_A$, $D_B$, as in the statement of Lemma 6.1.(1). Note that if one of $G$, $A$, $B$ is empty, then the statement holds trivially (with $C = \emptyset$, see Corollary 5.5). We now assume that $G$, $A$, $B$ are all non-empty and make a case analysis on the bottommost rule instance $r$ in $D$.

(a) The rule $r$ acts inside one of $G$, $A$, or $B$. I.e., the derivation $D$ is of shape

$$
\emptyset \quad \vdash D' \parallel \quad \emptyset
$$

for some $D'$. In each case we can apply the induction hypothesis to $D'$ as $|D'| < |D|$ and conclude immediately by adding the corresponding application of $r$ to $D_G$ or $D_A$ or $D_B$, respectively.

(b) $G = G' \mathrel{\mathcal{N}} G'$ and $D$ is of shape

$$
\emptyset \quad \vdash D' \parallel \quad \emptyset \quad \vdash D'' \parallel \quad \emptyset
$$

We make a case analysis on the structure of the graph $(A \otimes B)[G']_S$, using Lemma 3.7:

(i) $(A \otimes B)[G']_S$ cannot be the singleton graph as either $A$ nor $B$ are empty.

(ii) $(A \otimes B)[G']_S$ cannot be a par as this would imply $S = \emptyset$, contradicting the side condition of $ss \downarrow$.

(iii) $(A \otimes B)[G']_S$ is a tensor. We have the following possibilities:

(i) $(A \otimes B)[G']_S = A[G']_S \otimes B$ for some $S' \subseteq S$.

We can apply the induction hypothesis to $D'$, and get $C[\cdot]_R$ and $K'_A$ and $K_B$ such that

$$
C[K_A' \mathrel{\mathcal{N}} K_B']_{R'} \otimes G'' \mathrel{\mathcal{N}} (A \otimes B) \mathrel{\mathcal{N}} A[\cdot]_S \quad \text{and} \quad D_G \parallel D_A \parallel D_B \parallel C \parallel K_A' \parallel K' \mathrel{\mathcal{N}} A \mathrel{\mathcal{N}} A
$$

respectively.

(ii) $(A \otimes B)[G']_S = A \otimes B[G']_{S'}$ for some $S' \subseteq S$.

This case is similar to the previous one.

(iii) $(A \otimes B)[G']_S = A'' \otimes (A' \otimes B)[G']_{S'}$ for some $S' \subseteq S$, where $A = A' \otimes A''$ and $A'' \neq \emptyset \neq A'$.

We can apply the induction hypothesis to $D'$ and get $K''_A$ and $L$ and $C'[\cdot]_{R''}$ such that

$$
C'[K_A'' \mathrel{\mathcal{N}} L]_{R''} \otimes G'' \mathrel{\mathcal{N}} (A \otimes B) \mathrel{\mathcal{N}} A
$$

From $D_L$ we get that $\Gamma_{\text{GS}} G' \mathrel{\mathcal{N}} (A \otimes B)$ (via the rule $ss \downarrow$). To this we can apply the induction hypothesis to get a context $C''[\cdot]_{R''}$ and $K_A$ and $K_B$ such that

$$
C''[K_A'' \mathrel{\mathcal{N}} K_B'']_{R''} \otimes D_G \parallel D_C \parallel D_A \parallel D_B \parallel C'' \parallel K_A' \parallel K'' \parallel A \parallel A
$$
We can let $\mathcal{C}[:, R] = C'[C'\mathcal{C}'[:, R']]_{R'}$ (and hence $C = C'[C'\mathcal{C}'[:, R']]_R$), and $K_A = K_A'' \wedge K_A'$. We obtain $\mathcal{D}_G$ via

$$
\begin{array}{l}
\text{ss}1 \quad C'[\mathcal{C}'[K_A'' \wedge K_A' \wedge K_B][R']_R]
\end{array}
$$

and $\mathcal{D}_A$ via

Finally, $\mathcal{D}_C$ is obtained via Lemma A.2 from $\mathcal{D}_C'$ and $\mathcal{D}_C''$.

(iv) $(A \otimes B)[G']_S = (A \otimes B')[G']_S \otimes B''$ for some $S' \subseteq S$, where $B = B' \otimes B''$ and $B' \neq \emptyset \neq B''$.

This case is similar to the previous one.

(iv) $(A \otimes B)[G']_S$ is composed via a prime graph $Q$, i.e., it is of shape $Q\{A_1 \otimes B_1, \ldots, A_h \otimes B_h, G\}$ with $|V_Q| = h + 1 \geq 4$ and $A_1, \ldots, A_h$ being modules of $A$ and $B_1, \ldots, B_h$ being modules of $B$, such that at least one of $A_i, B_i$ is non-empty for every $i \in \{1, \ldots, h\}$. Note that we also have

$$Q(A_1 \otimes B_1, \ldots, A_h \otimes B_h, \emptyset) = A \otimes B \quad (30)$$

We apply the induction hypothesis to $\mathcal{D}'$ and get one of the following three cases

(a) We get $K_1, \ldots, K_h, K'_{G'}$ and a context $C'[\mathcal{C}']_R$, such that

$$C'[Q^+\{K_1, \ldots, K_h\}]_R$$

for all $1 \leq i \leq h$. Now observe that because of (30) we have the following proof

where $\hat{\mathcal{D}}$ is given by Lemma 5.2. To this proof we can apply the induction hypothesis (since $G'$ is non-empty), and get $K_A, K_B$ and a context $C'[\mathcal{C}']_R$, such that

$$C[K_A \wedge K_B]_R$$

Now $\mathcal{D}_G$ is the derivation

\[ C'[Q^+\{K_1, \ldots, K_h\}]_R \]

(\beta) We have $L_X$ and $L_Y$ and $C'[\mathcal{C}']_R$ such that

$$C'[L_X \wedge L_Y]_R$$

(Note that for easier typesetting we pick without loss of generality $A_1 \otimes B_1$, instead of an arbitrary $A_i \otimes B_i$ for some $i \in \{1, \ldots, h\}$.) From $\mathcal{D}_Y$, we get (via the rule $\text{ss}1$) that

$$t_{CS} L_Y \wedge G' \wedge Q(\emptyset, A_2 \otimes B_2, \ldots, A_h \otimes B_h, \emptyset) = A' \otimes B',$$

where $A = A'[A_1]_S$ and $B = B'[B_1]_T$.
for some $S, T$. We therefore have $K'_A$ and $K'_B$ and $C''[\cdot]_{R''}$ such that

$$C''[K'_A \uplus K'_B]_{R''}$$

Similarly, we can apply the induction hypothesis to $D_X$ and get have $K'_A$ and $K'_B$ and $C''[\cdot]_{R''}$ such that

$$C''[K'' \uplus A \uplus B]_{R''}$$

We let $K_A = K'' \uplus A \uplus B$, $K_B = K'' \uplus B$, and $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R''}$. Then $D_G$ is

$$C'[\cdot]_{R''}$$

Derivation $D_A$ is as follows:

$$\varnothing$$

and $D_B$ is similar. Finally, $D_C$ exists by Lemma A.2.

(y) We have $L_X$ and $L_Y$ and $C'[\cdot]_{R''}$ such that

$$C'[L_X \uplus L_Y]_{R''}$$

We apply the induction hypothesis to $D_Y$, using (30). This gives us $K_A$ and $K_B$ and $C''[\cdot]_{R''}$ such that

$$C''[K_A \uplus K_B]_{R''}$$

We let $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R''}$ and get $D_C$ from Lemma A.2. Finally, $D_G$ is

$$(c)$$

In the next case to consider we also have $G = G'' \uplus Y G'$, and $D$ is (without loss of generality) of shape

$$\varnothing$$

We proceed by a case analysis on $G'[A \uplus B]_S$, using Lemma 3.7. Observe that $G'[A \uplus B]_S$ cannot be a single-vertex graph, and without loss of generality, we can assume it is not of shape $E \uplus \emptyset D$ for non-empty $E, D$. Hence, there are only two cases to consider:

(iii) $G'[A \uplus B]_S$ is a tensor of two graphs. Without loss of generality, we can assume $G'[A \uplus B]_S = E \uplus D[A \uplus B]_{S'}$, with $E \neq \emptyset$. We apply the induction hypothesis to $D'$ and get $C'[\cdot]_{R''}$ and $K_E$ and $K_D$ such that

$$\varnothing$$

From $D_D$, we get $\varnothing K_D \uplus Y D \uplus (A \uplus B)$ via the $s\downarrow$-rule. Since $|E| > 0$ we can apply the induction hypothesis again, and get $C''[\cdot]_{R''}$ and $K_A$ and $K_B$ such that

$$\varnothing$$

Note that $G = G'' \uplus Y (E \uplus D)$. We therefore can let $C[\cdot]_R = C'[C''[\cdot]_{R''}]_{R''}$ and obtain $D_C$ from
Lemma A.2. We obtain $\mathcal{D}_C$ via

$$\frac{C'[\hat{K}]_{r''}}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)} \quad \frac{C'[K_D \mathcal{Y} (E \otimes D)]_{r'}}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)} \quad \frac{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}$$

where $\hat{K}$ abbreviates $K_A \mathcal{Y} K_B$.

(iv) $G'[A \otimes B]_{s'}$ is composed via a prime graph $Q$.
Without loss of generality, we can assume that $G'[A \otimes B]_{s'} = Q[N_1[A \otimes B]_{s'}], N_2, \ldots, N_m$ with $|V_Q| = m \geq 4$. We apply the induction hypothesis to $\mathcal{D}'$ and get one of the following three cases:

(iv.a) We have $C'[\cdot]_{r''}$ and $L_1, \ldots, L_m$ such that

$$\frac{C'[Q^i[L_1, \ldots, L_m]]_{r'}}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)} \quad \frac{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}$$

for $2 \leq i \leq m$. From $\mathcal{D}'_v$, we get (via the $\mathsf{ss}_\downarrow$-rule) that $r_{\mathcal{GS}} L_1 \mathcal{Y} N_1, \mathcal{Y} (A \otimes B)$, to which we can apply the induction hypothesis again and get $C''[\cdot]_{r''}$ and $K_A$ and $K_B$ such that

$$\frac{C''[K_A \mathcal{Y} K_B]_{r''}}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}$$

As $G = G'' \mathcal{Y} Q[N_1, \ldots, N_m]$, we can (as before) let $C[\cdot]_{r} = C'[C''[\cdot]_{r''}]_{r'}$ and obtain $\mathcal{D}_C$ as

$$\frac{C'[Q^i[L_1, \ldots, L_m]]_{r'}}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)} \quad \frac{C''[K_A \mathcal{Y} K_B]_{r''}}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}$$

i.e., we assume $i = 1$ in the second case of (2) in Lemma 6.1. From $\mathcal{D}'_v$ we get (via the $\mathsf{ss}_\downarrow$-rule) that $r_{\mathcal{GS}} L_X \mathcal{Y} N_1, \mathcal{Y} (A \otimes B)$, to which we can apply the induction hypothesis again to get $C''[\cdot]_{r''}$ and $K_A$ and $K_B$ such that

$$\frac{C''[K_A \mathcal{Y} K_B]_{r''}}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}$$

As before, we have $G = G'' \mathcal{Y} Q[N_1, \ldots, N_m]$. We let $C[\cdot]_{r} = C'[L_X Y, L_Y Q[N_1, N_2, \ldots, N_m]]_{r'}$, and obtain $\mathcal{D}_C$ as

$$\frac{C'[\mathcal{X} \mathcal{Y} Q[N_1, N_2, \ldots, N_m]]_{r'}}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}$$

where $\hat{K} = K_A \mathcal{Y} K_B$, and $\mathcal{D}_C$ is obtained via Lemma A.2 from $\mathcal{D}'_v$, $\mathcal{D}'$, and $\mathcal{D}_C$.

(iv.b) We have $C'[\cdot]_{r''}$ and $L_X$ and $L_Y$ such that

$$\frac{C'[L_X Y, L_Y Q[N_1, \ldots, N_m]]_{r'}}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}$$

i.e., we assume $i = 2$ in the second case of (2) in Lemma 6.1 (the cases $i \geq 3$ are similar). From $\mathcal{D}'_v$ we get $r_{\mathcal{GS}} L_X \mathcal{Y} Q[N_1, \mathcal{Y} N_2, \mathcal{Y} (A \otimes B)$, to which we can apply the induction hypothesis again, and get $C''[\cdot]_{r''}$ and $K_A$ and $K_B$ such that

$$\frac{C''[K_A \mathcal{Y} K_B]_{r''}}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}$$

$$\frac{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}{\mathcal{D}_C \vdash \mathcal{X} \mathcal{Y} (E \otimes D)}$$
As before, we have $G = G' \Downarrow (\mathcal{N}_1, \ldots, \mathcal{N}_m)$ and let $C[\cdot]_R = C'[C'[\cdot]_R]_R$. Thus, we can obtain $D_G$ as follows:

$$
C' \Downarrow (\mathcal{N}_1, \ldots, \mathcal{N}_m)
$$

where as before $\hat{K} = K_A \Downarrow K_B$ and $D_C$ is obtained via Lemma A.2.

(d) In the final case to consider we have that $G = G' \Downarrow \mathcal{Q}(\mathcal{N}_1, \ldots, \mathcal{N}_m)$ where $\mathcal{N}_i$ non-empty for all $i \in 1, \ldots, m$, and $\mathcal{Q}$ is prime with $|V_Q| \geq 4$, and $D$ is of shape

$$
G' \Downarrow (\mathcal{N}_1 \otimes (\mathcal{N}_2 \Downarrow (A_2 \otimes B_2)) \otimes \ldots \otimes (\mathcal{N}_m \Downarrow (A_m \otimes B_m)))
$$

This is only possible if

$$
A \otimes B = Q^+([\emptyset, A_2 \otimes B_2, \ldots, A_m \otimes B_m]) \quad (31)
$$

(observe at least one component of the prime connective $Q^+$ must be empty for this equality to hold and we take without loss of generality the first). We apply the induction hypothesis to $D'$ and get $C'[\cdot]_R$ and $K_1, \ldots, K_m$, such that

$$
C'[K_1 \Downarrow \ldots \Downarrow K_m]_R
$$

for all $i \in \{2, \ldots, m\}$. Then we can construct the following proof, by using (31):

$$
C'[\emptyset, K_2 \Downarrow \ldots \Downarrow K_m \Downarrow \mathcal{Q}(\mathcal{N}_1, \ldots, \mathcal{N}_m)]_R
$$

where $\hat{D}$ is given by Lemma 5.2. To this proof we can apply the induction hypothesis again (since $\mathcal{N}_1$ is non-empty), and get $C[\cdot]_R$ and $K_A$ and $K_B$, such that

$$
C[K_A \Downarrow K_B]_R
$$

It remains to give $D_G$ which is as follows:

$$
C'[\emptyset, K_2 \Downarrow \ldots \Downarrow K_m \Downarrow \mathcal{Q}(\mathcal{M}_1, \ldots, \mathcal{M}_n)]_R
$$

where $D_i$ consist of $m$ applications of the rule $s s 1$.  

(2) In this case, we assume $\tau_{GS} G \Downarrow P(M_1, \ldots, M_n)$ with $P$ prime and $|V_P| = n \geq 4$ and $M_i$ nonempty for $1 \leq i \leq n$; and aim to construct $C[\cdot]_R$, $D_G$, $D_C$, and either $K_1, D_1$ or $K_X, K_Y, D_X, D_Y$, as in the statement of the splitting lemma. As before, we make a case analysis on the bottommost rule instance $r$ in $D$.

(a) If the rule $r$ acts inside $G$ then we can conclude immediately by using the induction hypothesis. Similarly, if the rule $r$ acts inside one of the $M_i$, i.e., $D$ is of shape

$$
G \Downarrow P(M_1, \ldots, M_{i-1}, M_i', M_{i+1}, \ldots, M_n)
$$
for some $1 \leq i \leq n$, we can apply the induction hypothesis, unless $r$ is $a_{i\downarrow}$ and $M'_i = \emptyset$. Then we have

\[
\emptyset \vdash G \not\models P[M_1, \ldots, M_{i-1}, a\downarrow, \emptyset, M_{i+1}, \ldots, M_n]
\]

and can conclude immediately by letting $C = K_X = \emptyset$, and $K_Y = G$.

(b) $G = G' \not\models G''$ and $\mathcal{D}$ is of shape

\[
\emptyset \vdash G' \not\models P[M_1, \ldots, M_n][G']_{R_P}
\]

Now consider the possible forms of $P[M_1, \ldots, M_n][G']_{R_P}$, according to Lemma 3.7:

(i) It cannot be an atom since $G'$ and $M_i$ are non-empty.

(ii) It cannot be a par, due to conditions on $ss_1$.

(iii) It can only be of the form $P[M_1, \ldots, M_n] \otimes G'$.

In this case, we can apply the induction hypothesis to obtain $C'[-]_R$, $K_P$, $K''$ such that

\[
C'[K_P \not\models L'_G]_{R'} \vdash \emptyset, \quad \emptyset \not\models \emptyset, \quad \emptyset \not\models \emptyset, \quad \emptyset \not\models \emptyset.
\]

Using the above, we can construct the following derivation $\mathcal{D}_G$

Let $C[-]_R = C'[C'[-]_{R'}]_{R'}$ and obtain $\mathcal{D}_C$ by using Lemma A.2.

(iv) Otherwise, $P[M_1, \ldots, M_n][G']_{R_P}$ is a prime graph in which case we have the following possibilities.

(l) $\mathcal{D}$ is of the shape

\[
\emptyset \vdash G' \not\models P[M_1, \ldots, M_{j-1}, M_j[G']_S, M_{j+1}, \ldots, M_n]
\]

for some $1 \leq j \leq n$. We apply the induction hypothesis to $\mathcal{D'}$ and get one of the following three sub-cases:

(b.iv.i.a) We have $C[-]_R$ and $L_j$ and $K_1, \ldots, K_{j-1}, K_{j+1}, \ldots, K_n$ such that

\[
C[P^\perp(K_1, \ldots, K_{j-1}, L_j, K_{j+1}, \ldots, K_n)]_R \vdash \emptyset, \quad G''
\]

for $i \in \{1, \ldots, n\}$ with $i \neq j$. We let $K_j = L_j \not\models G'$. Then $\mathcal{D}_G$ is the derivation

\[
\emptyset \vdash \emptyset, \quad \emptyset \vdash \emptyset, \quad \emptyset \vdash \emptyset, \quad \emptyset \not\models \emptyset, \quad \emptyset \not\models \emptyset.
\]

or $\hat{K} = K_X \not\models K_Y$ for some $K_X$ and $K_Y$ such that

\[
\emptyset \vdash \emptyset, \quad \emptyset \not\models \emptyset.
\]

for some $j \in \{1, \ldots, n\}$.
(b.(iv).(I).\(\beta\)) We have \(C[-]_R\) and \(L_X\) and \(K_Y\) such that

\[
\begin{aligned}
&\frac{C[L_X \not\in K_Y]_R}{G''} , \\
&\frac{\emptyset}{D_X \parallel L_X \not\in M_j[G'']},
\end{aligned}
\]

We let \(K_X = L_X \not\in G'\) and let \(D_G\) and \(D_X\) be the derivations

\[
\begin{aligned}
&\frac{C[L_X \not\in K_Y]_R}{G''} , \\
&\frac{D_G \parallel L_X \not\in M_j[G'']}{G'},
\end{aligned}
\]

for some \(i \neq j\). We let \(K_Y = G' \not\in L_Y\), we obtain \(D_G\) as in the previous case, and \(D_X\) is

\[
\begin{aligned}
&\frac{\emptyset}{L_Y \not\in \{M_i, \ldots, M_i, \emptyset, M_{i+1}, \ldots, M_n\}},
\end{aligned}
\]

(II) \(D\) is of the shape

\[
\begin{aligned}
&\frac{\emptyset}{G'' \not\in Q(G', N_2, \ldots, N_k)},
\end{aligned}
\]

where \(Q\) is a prime graph such that \(|Q| = k > n\) such that

\[
Q(\emptyset, N_2, \ldots, N_k) = P(M_1, \ldots, M_n)
\]

and each \(N_i\) for \(i \in \{1, \ldots, k\}\) is a module of some \(M_j\) where \(j \in \{1, \ldots, n\}\).

By the induction hypothesis we have \(\hat{K}, C'[-]_R\) such that

\[
\begin{aligned}
&\frac{C'[-]_R}{D_X \parallel G'}
\end{aligned}
\]

And one of (a), (\(\beta\)) or (\(\gamma\)) holds as follows.
Now, since \( G' \) is nonempty the size of \( K'_Y \) \( \uparrow \) \( P(M_1, \ldots, M_n) \) is strictly less than the size of \( G \) \( \uparrow \) \( P(M_1, \ldots, M_n) \). Hence we can apply the induction hypothesis to obtain \( \mathcal{D} \) such that

\[
C'[\hat{K'}]_R' \\
\mathcal{D} \] 

\( Q^\uparrow(\emptyset, K_2, \ldots, K_k) \)

where \( \mathcal{D} \) satisfies most of the conditions of the splitting (providing \( K_i, K_X, K_Y, \) etc.). \( \mathcal{D}_G = C'[\hat{D}]_R' \circ \hat{D}'_R, C[\cdot]_R = C'[\cdot]_R \) and \( \mathcal{D}_C \) is given by Lemma A.2.

\((\gamma) \) \( \hat{K} = K'_X \uparrow K'_Y \) such that for some \( \ell \in \{2, \ldots, k\} \)

\[
\emptyset \quad \mathcal{D} \downarrow \]

\( K'_X \uparrow N_\ell \)

\( \emptyset \quad \mathcal{D} \downarrow \]

\( K'_Y \uparrow Q(G', N_2, \ldots, N_{\ell-1}, \emptyset, N_{\ell+1}, \ldots, N_k) \)

In the case that \( N_\ell = M_m \) for some \( m \in \{1, \ldots, k\} \), proof \( \mathcal{D}_Y \) is given by (using (32))

\[
\emptyset \quad \mathcal{D} \downarrow \]

\( K'_X \uparrow Q(G', N_2, \ldots, N_{\ell-1}, \emptyset, N_{\ell+1}, \ldots, N_k) \)

\( P(M_1, \ldots, M_{m-1}, M', M_{m+1}, \ldots, M_k) \)

\( \uparrow K'_Y \uparrow Y G' \uparrow P(M_1, \ldots, M_{m-1}, \emptyset, M_{m+1}, \ldots, M_k) \)

holds, using \( \mathcal{D}_Y \) and \( ss \), and is strictly smaller than \( G \uparrow P(M_1, \ldots, M_m) \), since \( N_j \) is non-empty. Hence we can apply the induction hypothesis to obtain \( \hat{K} \) and \( C'[\cdot]_R \) such that

\[
C'[\hat{K'}]_R' \\
\mathcal{D} \] 

\( [K'_Y, G'] \)

There are then three cases to check (A), (B) and (C) as follows.

(A) Let \( \hat{K}' = P^\uparrow (K_1, \ldots, L, K_{m+1}, \ldots, K_n) \) and

\[
\emptyset \quad \mathcal{D}_n \downarrow \]

\( L \uparrow M_i' \)

\( \emptyset \quad \mathcal{D}_i \downarrow \]

\( K_i \uparrow M_i \)

for \( i \in \{1, \ldots, m-1\} \cup \{m+1, \ldots, n\} \).

From the above we can construct the following derivation \( \mathcal{D}_G \)

\[
\emptyset \quad \mathcal{D} \downarrow \]

\( C'[P^\uparrow (K_1, \ldots, K'_X \uparrow L, K_{m+1}, \ldots, K_n)]_R' \)

\( ss \) \[ \]

\( C'[K'_X \uparrow K'_Y]_R' \)

\( \uparrow G'' \)

\( \uparrow G' \)

Also we can construct proof \( \mathcal{D}_m \) as follows

\[
\emptyset \quad \mathcal{D} \downarrow \]

\( L \uparrow M'[\mathcal{D}_m \downarrow] \)

\( \emptyset \quad \mathcal{D}_m \downarrow \]

\( K'_X \uparrow G' \)

\( \uparrow N_j \)

\( \emptyset \quad \mathcal{D}_m \downarrow \]

\( K'_X \uparrow Y L \uparrow M'[N_j]_R_m \)

We conclude by setting \( K_m = K'_X; L \) and \( C[\cdot]_R = C'[\cdot]_R \), where \( \mathcal{D}_C \) is obtained using Lemma A.2.

\((B) \) Let \( \hat{K}' = K'_X \uparrow K'_Y \) where w.l.o.g.

\[
\emptyset \quad \mathcal{D}_X \downarrow \]

\( K_Y \uparrow Y P(\emptyset, \ldots, M_{m-1}, M', M_{m+1}, \ldots, M_n) \)

In this case we have derivation \( \mathcal{D}_G \), defined as follows

\[
\emptyset \quad \mathcal{D} \downarrow \]

\( C'[K'_X \uparrow K'_Y \uparrow K_Y]_R' \)

\( ss \) \[ \]

\( C'[K'_X \uparrow K'_Y]_R' \)

\( \uparrow G'' \)

\( \uparrow G' \)
We also have the following proof, named $\mathcal{D}_Y$

\[
\begin{array}{c}
\emptyset \\
K_Y \not\vdash P(\emptyset, \ldots, M_{m-1}, M_m) & |_{R_m, M_{m+1}, \ldots, M_n} \\
\emptyset \\
K_Y \not\vdash P(\emptyset, \ldots, M_{m-1}, \emptyset, M_{m+1}, \ldots, M_n) & |_{R_m, M_{m+1}, \ldots, M_n}
\end{array}
\]

We conclude by setting $K_Y = K'_Y \not\vdash K_Y$ and $C[\cdot]_R = C'[C''[\cdot]_{R'}]_{R'}$, where $\mathcal{D}_C$ is obtained using Lemma A.2.

(c) Let $K' = K'_Y \not\vdash K_Y$ where w.l.o.g.

\[
\emptyset \\
K_X \not\vdash M' \\
\emptyset \\
K_Y \not\vdash P(\emptyset, \ldots, M_{m-1}, \emptyset, M_{m+1}, \ldots, M_n)
\]

In this case we have derivation $\mathcal{D}_G$, defined as follows

\[
\begin{array}{c}
\emptyset \\
C'[K_X \not\vdash K_Y]_{R'} \\
\emptyset \\
C'[K_Y \not\vdash G']
\end{array}
\]

We also have the following proof, named $\mathcal{D}_X$

\[
\begin{array}{c}
\emptyset \\
K_X \not\vdash P(\emptyset, \ldots, M_{m-1}, M_m) & |_{R_m} \\
\emptyset \\
K_X \not\vdash P(\emptyset, \ldots, M_{m-1}, \emptyset, M_{m+1}, \ldots, M_n) & |_{R_m}
\end{array}
\]

We conclude by setting $K_X = K'_X \not\vdash K_X$ and $C[\cdot]_R = C'[C''[\cdot]_{R'}]_{R'}$, where $\mathcal{D}_C$ is obtained using Lemma A.2.

(c) If $G = G'' \not\vdash G'$ (with $G' \neq \emptyset$) and $\mathcal{D}$ is of shape

\[
\begin{array}{c}
\emptyset \\
G'' \not\vdash G'[P(M_1, \ldots, M_n)]_S \\
G'' \not\vdash G' \not\vdash P[M_1, \ldots, M_n]
\end{array}
\]

we proceed as in case (1.c) by a case analysis on the shape of $G'[P(M_1, \ldots, M_n)]_S$ via Lemma 3.7, and for the same reasons as above, there are two cases.

(iii) $G'[P(M_1, \ldots, M_n)]_S$ is a tensor of two graphs. Without loss of generality, we can assume $G'[P(M_1, \ldots, M_n)]_S = E \otimes D[P(M_1, \ldots, M_n)]_S$, where $E \neq \emptyset$. We apply the induction hypothesis to $D'$ and get $C'[\cdot]_{R'}$ and $K_E$ and $K_D$ such that

\[
\begin{array}{c}
\emptyset \\
C'[K_E \not\vdash K_D]_{R'} \\
\emptyset \\
C'
\end{array}
\]

From $\mathcal{D}_D$, we get $\mathcal{D}_G$, $K_D \not\vdash D \not\vdash P(M_1, \ldots, M_n)$ via the ss|-rule. Since $|E| > 0$ we can apply the induction hypothesis again, giving us one of the following two cases:

- either a context $C'[\cdot]_{R'}$ and graphs $K_1, \ldots, K_n$, such that

\[
\begin{array}{c}
\emptyset \\
C'[P^2(K_1, \ldots, K_n)]_{R'} \\
\emptyset \\
C' \not\vdash C'
\end{array}
\]

for all $i \in \{1, \ldots, n\}$,

- or a context $C'[\cdot]_{R'}$ and graphs $K_X$ and $K_Y$ such that

\[
\begin{array}{c}
\emptyset \\
C'[K_X \not\vdash K_Y]_{R'} \\
\emptyset \\
C' \not\vdash C'
\end{array}
\]

for some $i \in \{1, \ldots, n\}$.

In the first case we let $K = P^i(K_1, \ldots, K_n)$ and in the second $K = K_X \not\vdash K_Y$. In both cases we have $G = G'' \not\vdash (E \otimes D)$ and we can let $C[\cdot]_R$ and $\mathcal{D}_G$ as in case (1.c.iii) above.⁶

(iv) $G'[P(M_1, \ldots, M_n)]_S$ is a tensor of a prime graph $Q$. This case is similar to case (1.c.iv) above. Without loss of generality, we can assume that $G'[P(M_1, \ldots, M_n)]_S = Q[N_1(P[M_1, \ldots, M_n])_S, N_2, \ldots, N_m]$ for some prime graph $Q$ and $m \geq 4$, and $N_i \neq \emptyset$ for all $1 \leq i \leq m$. Applying the induction hypothesis to $D'$ gives us one of the following three cases:

(vi.α) We have $C'[\cdot]_{R'}$ and $L_1, \ldots, L_m$ such that

\[
\begin{array}{c}
\emptyset \\
C'[Q^2(L_1, \ldots, L_m)]_{R'} \\
\emptyset \\
C'
\end{array}
\]

⁶This is the reason for using the abbreviation $K$ in that case.
for $2 \leq i \leq m$. From $\mathcal{D}'$, we get (via the rule ss) $\der{\mathcal{D}_r}{\mathcal{D}_c}{\mathcal{D}_g}$, $L_1 \not\models N_1 \P M_1, \ldots, M_n)_{i'}$ to which we can apply the induction hypothesis again, to get one of the following two cases:

- either a context $C''[]_{R'}$ and graphs $K_1, \ldots, K_n$, such that

$$C''[P^+(K_1, \ldots, K_n)]_{R'} \der{L_1 \not\models N_1 \P \mathcal{D}_r}{\mathcal{D}_c}{\mathcal{D}_g} \der{L_1 \not\models N_1 \P \mathcal{D}_r}{\mathcal{D}_c}{\mathcal{D}_g}$$

for all $i \in \{1, \ldots, n\}$,

- or a context $C''[]_{R'}$ and graphs $K_X$ and $K_Y$ such that

$$C''[K_X \not\models K_Y]_{R'} \der{L_1 \not\models N_1 \P \mathcal{D}_r}{\mathcal{D}_c}{\mathcal{D}_g}$$

for some $i \in \{1, \ldots, n\}$.

As above, we let in the first case $\hat{K} = P^+(K_1, \ldots, K_n)$ and in the second $\hat{K} = K_X \not\models K_Y$. In both cases we have $G = G'' \P Q[N_1, \ldots, N_m]$ and can let $C[]_{R'}$ and $\mathcal{D}_g$ and $\mathcal{D}_c$ as in case (1.c.iv) above.

(iv.β) We have $C''[]_{R'}$ and $L_X$ and $L_Y$ such that

$$C''[L_X \not\models L_Y]_{R'} \der{L_X \not\models N_1 \P \mathcal{D}_r}{\mathcal{D}_c}{\mathcal{D}_g}$$

From $\mathcal{D}_X$ we get $\der{\mathcal{D}_r}{\mathcal{D}_c}{\mathcal{D}_g}$, $L_X \not\models N_1 \P M_1, \ldots, M_n)$, to which we apply the induction hypothesis again and get one of the following two cases:

- either a context $C''[]_{R'}$ and graphs $K_1, \ldots, K_n$, such that

$$C''[P^+(K_1, \ldots, K_n)]_{R'} \der{L_X \not\models N_1 \P \mathcal{D}_r}{\mathcal{D}_c}{\mathcal{D}_g}$$

for all $i \in \{1, \ldots, n\}$,

- or a context $C''[]_{R'}$ and graphs $K_X$ and $K_Y$ such that

$$C''[K_X \not\models K_Y]_{R'} \der{L_X \not\models N_1 \P \mathcal{D}_r}{\mathcal{D}_c}{\mathcal{D}_g}$$

for some $i \in \{1, \ldots, n\}$.

As above, we let in the first case $\hat{K} = P^+(K_1, \ldots, K_n)$ and in the second $\hat{K} = K_X \not\models K_Y$. In both cases we have $G = G'' \P Q[N_1, \ldots, N_m]$ and can let $C[]_{R'}$ and $\mathcal{D}_g$ and $\mathcal{D}_c$ as in case (1.c.iv) above.

---

7This is the reason for using the abbreviation $\hat{K}$ in that case.
(d) Consider when we have $G = G' \not\in \mathcal{P}(N_1, \ldots, N_n)$ and $\mathcal{D}$ is of shape

$$
\begin{align*}
\varnothing & \vdash G' \\
\mathcal{D}' \parallel \ldots \parallel (N_1 \not\in M_1) \otimes \cdots \otimes (N_n \not\in M_n) \\
\mathcal{P} & \parallel G' \\
\mathcal{D}' \parallel (N_1 \not\in M_1) \otimes \cdots \otimes (N_n \not\in M_n)
\end{align*}
$$

By applying the induction hypothesis $n - 1$ times to $\mathcal{D}'$ and some uses of the \text{ss} rule, we obtain a context $C[\cdot]_R$ and graphs $L_1, \ldots, L_n$ such that

$$
C[L_1 \not\in \cdots \not\in L_n]_R
\vdash \begin{array}{c}
\varnothing \\
\mathcal{D}_C \parallel \varnothing \\
\mathcal{D}_C \parallel C \\
L_1 \not\in N_1 \not\in M_1
\end{array}
$$

for $i \in \{1, \ldots, n\}$. We let $K_i = L_i \not\in N_i$, and construct $\mathcal{D}_G$ as

$$
\begin{align*}
C[L_1 \not\in \cdots \not\in L_n]_R & \vdash \varnothing \\
\mathcal{D}_G \parallel \varnothing \\
\mathcal{D}_G \parallel C \\
L_1 \not\in \cdots \not\in L_n
\end{align*}
$$

(e) Finally, consider when $G = G'' \not\in Q(N_1, \ldots, N_k)$ where $N_i$ non-empty for all $j, Q$ is prime. $|V_Q| > |V_P|$, $P(M_1, \ldots, M_n) = Q'(\emptyset, L_2, \ldots, L_n)$ (observe at least one module of the prime connective $Q'$ must be empty for this equality to hold and we set w.l.o.g. to be the first module, otherwise $P'$ and $Q$ are isomorphic, contradicting $|V_Q| > |V_P|$), and $\mathcal{D}$ is of shape

$$
\begin{align*}
\varnothing & \vdash \mathcal{D}' \parallel \ldots \parallel \mathcal{D}' \\
G'' & \parallel (N_1 \not\in (N_2 \not\in L_2) \otimes \cdots \otimes (N_n \not\in L_n)) \\
G'' & \parallel Q(N_1, \ldots, N_k) \not\in P(M_1, \ldots, M_n)
\end{align*}
$$

By applying the induction hypothesis to $\mathcal{D}'$ we can obtain $C''[\cdot]_R$ and $K_i$ such that

$$
\begin{align*}
C''[K_1; K_2; \ldots; K_n]_R & \vdash \varnothing \\
\mathcal{D}_G \parallel \varnothing \\
\mathcal{D}_G \parallel C \\
N_1 \not\in K_1 \not\in K_1 \not\in M_i
\end{align*}
$$

Now observe we can construct the following proof, where $\mathcal{D}_Q$ is given by Lemma 5.2,
(d) Assume $P$ is prime, $|V_P| \geq 4$, and $M_i$ are nonempty in the derivation

\[
\begin{array}{c}
\varnothing \\
P \vdash \emptyset \quad \emptyset \\
G'' \cap ((a \cap M_1) \otimes M_2 \otimes \ldots \otimes M_n) \\
G'' \cap P[M_1, \ldots, M_n] \cap \cap a
\end{array}
\]

Notice, w.l.o.g., $a$ is ready to interact with the first module of the $P$ connective. By the induction hypothesis, we have $C'[\cdot|_R$, $K_i$ such that

\[
C'[K_1 \cap \ldots \cap K_n]|_R, \quad D_C \cap C', \quad \emptyset \cap D_i \cap K_i \cap M_i
\]

for $2 \leq i \leq n$. Since $+ K_1 \cap M_1 \cap \cap a$, by the induction hypothesis, we have $C'[\cdot|_R'$ such that

\[
C'[a_1]|_R, \quad D_i \cap K_i \cap M_i
\]

From the above, we can construct the following derivation, $D_C$, as required, where $C[\cdot|_R = C'[\cdot|_R']$, and $D_C$ is given by Lemma A.2

(e) In the final case we have $G = G' \cap a^\bot$ and $D$ is of shape

\[
\begin{array}{c}
\emptyset \\
D_G \cap \emptyset \\
G' \cap a^\bot \cap \cap a
\end{array}
\]

We can immediately conclude by letting $C = \emptyset$ (hence $D_C$ is trivial) and letting $D_G$ be

\[
D_G \cap \emptyset \quad \emptyset \cap G'
\]

This completes the proof of the splitting lemma. $\square$

## B Proof of Context Reduction (Lemma 7.1)

**Proof of Lemma 7.1.** We proceed by induction on the size of $G[A]_S$, making a case analysis based on Lemma 3.7.

(i) $G[A]_S$ has only one vertex. This is impossible as it is provable.

(ii) $G[A]_S = G'' \cap G'[A]_S$: In that case we make the same case analysis on $G'[A]_S$. But without loss of generality, we can assume that $G'[A]_S$ is not a par.

(ii) $G'[A]_S$ has only one vertex. Then $G'[A]_S = A = a$ for some atom and $G[A]_S = G'' \cap a$. We apply Lemma 6.1.(3) and obtain $C[\cdot|_R$ such that

\[
C[a_1]|_R, \quad D'_G \cap \emptyset \quad \emptyset \cap G''
\]

We let $K = a^\bot$ and for any $X$, we can construct

\[
C[a^\bot \cap X]|_R, \quad D'_G \cap \emptyset \quad \emptyset \cap G''
\]

as $G[X]_S = G'' \cap X$.

(ii) $G'[A]_S$ is a tensor. Then we can assume without loss of generality that $G[A]_S = G'' \cap G'[A]_S \otimes G_2$. By Lemma 6.1.(1), we get $C[\cdot|_R'$ and $K_1$ and $K_2$, such that

\[
C'[K_1 \cap K_2]|_R', \quad D_C \cap \emptyset \quad \emptyset \cap G''
\]

We apply the induction hypothesis to $D_1$ and get $K$ and $C'[\cdot|_R'$, such that

\[
\begin{array}{c}
\emptyset \cap D'_C \cap C', \quad \emptyset \cap D_i \cap K_i \cap G'[A]_S, \quad \emptyset \cap D_i \cap K_i \cap G_2
\end{array}
\]

for any $X$. We now let $C[\cdot|_R = C'[\cdot|_R']$ and $D_G$ as

\[
\begin{array}{c}
C'[K \cap X]|_R', \quad D'_G \cap \emptyset \quad \emptyset \cap G''
\end{array}
\]

We obtain $D_C$ from Lemma A.2 and construct $D_G$ as

\[
\begin{array}{c}
C'[K_1 \cap K_2]|_R', \quad D'_G \cap \emptyset \quad \emptyset \cap G''
\end{array}
\]

since $G[X]_S = G'' \cap (G'[X]_S \otimes G_2)$.
(ii.iv) \(G'[A]_S\) is composed via a prime graph \(P\) with \(|V_P| \geq 4\). Then, without loss of generality we can assume that \(G[A]_S = G'' \nparallel P[M_1[A]_S, M_2, \ldots, M_n]\).

Applying Lemma 6.1.2 gives us one of the following three cases:

(a) We have \(C'[\cdot ]_R\) and \(K_1, \ldots, K_n\), such that

\[
\begin{align*}
&\alpha \\
&\beta \\
&\gamma \\
&\delta
\end{align*}
\]

\(\phi\)

for \(2 \leq i \leq n\). We can apply the induction hypothesis to \(D_i\) and obtain \(K\) and \(C''[\cdot ]_{R''}\), such that

\[
\begin{align*}
&\phi \\
&\psi \\
&\chi \\
&\epsilon
\end{align*}
\]

for any \(X\). We let \(C[\cdot ]_R\) and \(D_C\) as in case (ii.iii) above and obtain \(D_G\) as

\[
\begin{align*}
&\phi' \\
&\psi' \\
&\chi' \\
&\epsilon'
\end{align*}
\]

where \(G[X]_S = G'' \nparallel P[M_1[X]_S, M_2, \ldots, M_n]\).

(b) We have \(C'[\cdot ]_R\) and \(K_X\) and \(K_Y\), such that

\[
\begin{align*}
&\phi \\
&\psi \\
&\chi \\
&\epsilon
\end{align*}
\]

We apply the induction hypothesis to \(D_X\) and get \(K\) and \(C''[\cdot ]_{R''}\), such that

\[
\begin{align*}
&\phi' \\
&\psi' \\
&\chi' \\
&\epsilon'
\end{align*}
\]

for any \(X\). We let \(C[\cdot ]_R\) and \(D_C\) as in case (ii.iii) above and obtain \(D_G\) as

\[
\begin{align*}
&\phi'' \\
&\psi'' \\
&\chi'' \\
&\epsilon''
\end{align*}
\]

(iii) \(G[A]_S\) is a tensor. This case is as case (ii.iii) above, with \(G'' = \emptyset\), and consequently \(C' = K_1 = K_2 = \emptyset\).

(iv) \(G[A]_S\) is composed via a prime graph \(P\) with \(|V_P| \geq 4\).

This case is as case (ii.iv) above, with \(G'' = \emptyset\). Consequently, \(C'\) and \(K_1, \ldots, K_n\) (resp. \(K_X\) and \(K_Y\)) are empty as well.

\(\Box\)

C Proof of admissibility of the up-rules (Section 8)

In this section we show how splitting and context reduction are used to show the admissibility of all rules in the up-fragment of SGS.

**Theorem C.1.** The rule ai is admissible for GS.
Proof. Assume we have a proof of $G[a \otimes a^+]_S$. By Lemma 7.1 we have a graph $L$ and a context $C_1[\_]_{R_1}$, such that there are derivations

\[
\emptyset \vdash_1 \emptyset \ G[L \otimes (a \otimes a^-)]_S, \quad \emptyset \vdash_1 \emptyset \ L \otimes X \ G[X]_S
\]

for any graph $X$. We apply Lemma 6.1.(1) to $D_2$ and get $K_a$ and $K_a^+$ and a context $C_2[\_]_{R_2}$ such that

\[
C_2[K_a \otimes K_a^+]_{R_2} \vdash_1 \emptyset \ G[L]_S, \quad C_2 \vdash_1 \emptyset \ G[C_3]_S, \quad C_2 \vdash_1 \emptyset \ G[C_4]_S
\]

Applying Lemma 6.1.(3) to $D_5$ and $D_7$ gives us $C_3[\_]_{R_3}$ and $C_4[\_]_{R_4}$ such that

\[
C_3[a^+]_{R_3} \vdash_1 \emptyset \ G[K_a]_S, \quad C_3 \vdash_1 \emptyset \ G[C_3]_S, \quad C_4[a]_{R_4} \vdash_1 \emptyset \ G[K_a^+]_S, \quad C_4 \vdash_1 \emptyset \ G[C_4]_S
\]

We can now give the following derivation

that proves $G = G[\_]_S$ in GS.

\[\square\]

Theorem C.2. The rule $\text{ss}^+ \vdash_1$ is admissible for GS.

Proof. Assume we have a proof of $G[B \otimes A]_S$ in GS. By Lemma 7.1 we have a graph $L$ and a context $C_1[\_]_{R_1}$, such that there are derivations

\[
\emptyset \vdash_1 \emptyset \ G[L \otimes (B \otimes A)]_S, \quad \emptyset \vdash_1 \emptyset \ L \otimes X \ G[X]_S
\]

for any graph $X$. We apply Lemma 6.1.(1) to $D_2$ and get $K_B$ and $K_A$ and a context $C_2[\_]_{R_2}$ such that

\[
C_2[K_B \otimes K_A]_{R_2} \vdash_1 \emptyset \ G[L]_S, \quad C_2 \vdash_1 \emptyset \ G[C_3]_S, \quad C_2 \vdash_1 \emptyset \ G[C_4]_S
\]

We can now give a proof of $G[B[A]_T]_S$ as follows:

The proof of the next theorem is different from the others, and also different from what usually happens in a deep inference cut elimination proof. Even though we still use splitting and context reduction, we additionally need an induction on the size of the cut.

Theorem C.3. The rule $p \uparrow \vdash_1$ is admissible for GS.

Proof. We define the size of an instance of $p \uparrow$ (see Figure 1) as

\[
\sum_{1 \leq i \leq n} (|M_i| + |N_i|)
\]

i.e., the number of vertices in the subgraph that are modified by the rule. We now proceed as in the previous two proofs. Assume we have a proof of $G(P[M_1, \ldots, M_n] \otimes P^+[N_1, \ldots, N_n])_S$. We apply Lemma 7.1 and get a graph $L$ and a context $C_1[\_]_{R_1}$, such that there are derivations

\[
\emptyset \vdash_1 \emptyset \ G[L \otimes X]_S, \quad \emptyset \vdash_1 \emptyset \ G[X]_S
\]

for any $\emptyset \subseteq T \subset |V_B|$. \[\square\]
Applying Lemma 6.1.(2) to $D_6$ and $D_7$ gives us four different cases.

(a) We get $K_1, \ldots, K_n$ and $H_1, \ldots, H_n$ and contexts $C_3[\cdot]_{R_i}$ and $C_4[\cdot]_{R_i}$, such that

$$C_3[P^i(K_1, \ldots, K_n)]_{R_i} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset}$$

for all $i, j \in \{1, \ldots, n\}$. Then, our proof of $G(M_1 \otimes N_1) \not\rightarrow \cdots \not\rightarrow (M_n \otimes N_n)_S$ is shown in Figure 4, where $D^*$ exist by Lemma 5.2.

(b) We get $K_1, \ldots, K_n$ and $H_X$ and $H_Y$ and contexts $C_3[\cdot]_{R_i}$ and $C_4[\cdot]_{R_i}$, such that

$$C_3[H_X \not\rightarrow H_Y]_{R_i} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset}$$

for all $i \in \{1, \ldots, n\}$ and some $j \in \{1, \ldots, n\}$. The derivation for $G(M_1 \otimes N_1) \not\rightarrow \cdots \not\rightarrow (M_n \otimes N_n)_S$ is shown in Figure 3, where $D^*$ consists of $n+1$ instances of $SS$ and the double-line in the center indicates two instances of $SS$. The only instance of $K \not\rightarrow 1$ occurs as bottommost rule instance in $D^*$.

(c) We get $K_Z$ and $K_W$ and $H_1, \ldots, H_n$ and contexts $C_3[\cdot]_{R_i}$ and $C_4[\cdot]_{R_i}$, such that

$$C_3[K_Z \not\rightarrow K_W]_{R_i} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset}$$

for some $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, n\}$. This case is similar to the previous one.

(d) We get $K_Z$ and $K_W$ and $H_X$ and $H_Y$ and contexts $C_3[\cdot]_{R_i}$ and $C_4[\cdot]_{R_i}$, such that

$$C_3[K_Z \not\rightarrow K_W]_{R_i} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset} \quad \frac{\emptyset}{\emptyset}$$

for some $i, j \in \{1, \ldots, n\}$. In this case we use the derivation in Figure 6 to prove $G(M_1 \otimes N_1) \not\rightarrow \cdots \not\rightarrow (M_n \otimes N_n)_S$. More precisely, Figure 6 shows the case $i < j$, the cases $i = j$ and $i > j$ are similar. The derivation $D^*$ exist by the second statement in Lemma 5.2. If $i \neq j$, this derivation consists of a single $\not\rightarrow$ instance. If $i = j$, it can be a longer derivation containing all rules of SGS (where the instances of $a^i$ and $ss^j$ can be eliminated by the previous two theorems). The important observation to make is that all instances of $p^i$ occurring in $D^*$ have smaller size than the one we started with. Therefore we can invoke the induction hypothesis. \hfill \square

D Proof of Conservativity (Lemma 9.1)

Proof of Lemma 9.1. By way of contradiction, assume there is a cograph that is not provable with passing through a non-cograph. Let $A$ be a minimal such graph, where we define the size of $A$ as the lexicographic pair $|V_A|, |E_A|$. The only way to create a non-cograph from a cograph while going up in a derivation is via the $SS$ as in

$$\frac{P(M_1, \ldots, M_i, M_{i+1}, \ldots, M_n)}{\not\rightarrow M_i \not\rightarrow P(M_1, \ldots, M_{i+1}, \ldots, M_n)}$$

where $M_i$ and $P(M_1, \ldots, M_i, \emptyset, M_{i+1}, \ldots, M_n)$ are cographs and $P(M_1, \ldots, M_n)$ is not. Without loss of generality, we assume $i = 1$. By minimality of $A$, we can assume that this $SS$ occurs as bottommost rule instance in $D$, and we can also assume that it occurs in a shallow context, i.e., we have

$$A = G \not\rightarrow M_1 \not\rightarrow P(\emptyset, M_2, \ldots, M_n)$$

for some $G$. Otherwise $A = G \not\rightarrow C[M_1 \not\rightarrow P(\emptyset, M_2, \ldots, M_n)]_R$ for some nontrivial context $C[\cdot]_R$, and we could apply context reduction to get a $K$ with $\not\rightarrow K \not\rightarrow P(\emptyset, M_2, \ldots, M_n)$.
Figure 4. Derivation for case (a) in the proof of Theorem C.3

Figure 5. Derivation for case (b) in the proof of Theorem C.3
contradicting the minimality of \( A \). Hence, \( D \) is of shape

\[
\begin{array}{c}
\emptyset \\
\emptyset \vdash \emptyset \\
\emptyset \vdash \emptyset \\
\emptyset \vdash \emptyset \\
\emptyset \vdash \emptyset
\end{array}
\]

and we apply splitting (Lemma 6.1.2) to \( D' \), yielding 2 possibilities, of which we show here only the first, the second one being simpler.

- there is a context \( C[\cdot]_R \) and graphs \( K_1, \ldots, K_n \), such that

\[
C[P^i(K_1, \ldots, K_n)]_R \\
\emptyset \vdash \emptyset \\
\emptyset \vdash \emptyset \\
\emptyset \vdash \emptyset
\]

for all \( i \in \{1, \ldots, n\} \). If \( |D_G| = 0 \) then \( G = C[P^i(K_1, \ldots, K_n)]_R \) and we can have a derivation

\[
C[P^i(K_1, \ldots, K_n)]_R \\
\emptyset \vdash \emptyset \\
\emptyset \vdash \emptyset \\
\emptyset \vdash \emptyset
\]

contradicting the minimality of \( A \). If \( |D_G| \neq 0 \) and there is a cograph \( G' \) occurring in \( D_G \), then \( G' \) has smaller size than \( G \), contradicting the minimality of \( G \). If there is no smaller cograph \( G' \) occurring in \( D_G \) in, then the bottom-most rule instance in \( D_G \) is a \( \text{ss}_\downarrow \) creating a non-cograph. By the same argument as above, we can conclude, that it must be in a shallow context. Then \( G = K_i \ interpreted in \( D_G \), such that if \( i = 1 \) we apply \( \text{ss}_\downarrow \) to move \( M_i \) and \( K_i \) inside \( C[\cdot]_R \) and conclude by a similar reasoning as with (33). If \( i \neq 1 \) we have \( P^i(K_1, \ldots, K_{i-1}, \emptyset, K_{i+1}, \ldots, K_n) \) and

\[
\begin{array}{c}
K_W \not\vdash \emptyset \\
H_Y \not\vdash \emptyset
\end{array}
\]

Figure 6. Derivation for case (d) in the proof of Theorem C.3.
$P(\emptyset, M_2, \ldots, M_n)$ are cographs and again we get a contradiction to the minimality of $A$. □