Mechanism Design (with and) without Money

“Algorithmic Game Theory”, Ch. 9-10
“A Course in Game Theory”, Ch. 10
“A Primer in Social Choice Theory”, Ch. 5
Outline

- Social Choice Theory
- Implementation Theory
- Mechanism Design With Money
- Mechanism Design Without Money
Social Choice Theory
Collective decision-making

- Elections
- Auctions
- Program Committees
Social Choice

A Social Choice structure is a quadruple:

\[ \mathcal{G} = \langle \text{Agn}, \text{Iss}, \text{Prf}, \text{Sc} \rangle \]

s.t:

- \( \text{Agn} \) is a finite set of agents such that \( 1 \leq |\text{Agn}| \);
- \( \text{Iss} \) is a finite set of issues such that \( 3 \leq |\text{Iss}| \);
- \( \text{Prf} \) is the set of all preference profiles, i.e., \(|\text{Agn}|\)-tuples \( p = (\preceq_i)_{i \in \text{Agn}} \) where each \( \preceq_i \) is a total order over \( \text{Iss} \);
- \( \text{Sc} \) is a function taking each \( p \in \text{Prf} \) to an element in \( \text{Iss} \), i.e.:

\[ \text{Sc} : \text{Prf} \rightarrow \text{Iss} \]
What classes of Social Choice functions are possible?

<table>
<thead>
<tr>
<th>Input</th>
<th>Function</th>
<th>Output</th>
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<td>$\prec_1$</td>
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Incentive-compatibility

A social choice function $\mathcal{S}c$ can be strategically manipulated by agent $i$ if for some profile $p = (\succeq_1, \ldots, \succeq_n)$, there exists another profile $p' = (\succeq_{-i}, \succeq'_i)$ s.t.

$$\mathcal{S}c(p) \preceq_i \mathcal{S}c(p')$$

for $\mathcal{S}c(p) \neq \mathcal{S}c(p')$. A function $\mathcal{S}c$ is *incentive compatible* (or strategy-proof) if it cannot be manipulated.

- An agent can force a different alternative which he prefers by misrepresenting his preferences
- Majority voting on a set of issues with 2 elements is strategy-proof
Why incentive compatibility?

- If we want to construct a social choice function as an *algorithm* we have, first of all, to elicit the preferences of the agents
- Preferences are private
- Incentive compatibility guarantees that our algorithm elicits the right information
Dictatorship

A social choice function $S_c$ is a dictatorship if there exists an agent $i$ s.t. $\forall \mathbf{p} \in \text{Prf}$, and $\forall b \neq a \in \text{Iss}$:

$$ b \preceq_i a \Rightarrow S_c(\mathbf{p}) = a $$

A function $S_c$ is non-dictatorial if there is no dictator.

- Is a dictatorship incentive compatible?
Gibbard-Satterthwaite Theorem ('73, '75)

Gibbard-Satterthwaite Theorem ('73, '75)

Let \( \text{Iss} > 2 \). If a social choice function \( \mathcal{Sc} \) is:

1. onto \( \text{Iss} \) (aka non-imposition) and
2. is incentive compatible

then it is a dictatorship.

- How does this jeopardize the possibility of finding algorithms for collective decision-making?
Implementation
Implementation

An Implementation Problem for the Social Choice structure $\mathcal{S}$ is a structure:

$$(\mathcal{S}, \mathcal{G})$$

where $\mathcal{G}$ is a set of strategic game forms $G = (\text{Agn}, \text{Str}, g)$ s.t:

- $\text{Agn}$ is the finite set of agents of $\mathcal{S}$;
- $\text{Str}$ is the set $\prod_{i \in \text{Agn}} S_i$ of all strategy profiles, where $S_i$ is the set of strategies of agent $i$;
- $g$ is the outcome function of the game: $g : \text{Str} \rightarrow \text{Iss}$

Given $(\mathcal{S}, \mathcal{G})$, find a game $G \in \mathcal{G}$ and a solution concept $S$ s.t.:

$$g(S(G, p)) = Sc(p)$$

If such a $G$ exists then $Sc$ is $S$-implementable.
**Truthful implementation**

A Direct Implementation Problem for the Social Choice structure \( \mathcal{S} \) is a structure:

\[(\mathcal{S}, \mathcal{G})\]

where \( \mathcal{G} \) is a set of strategic game forms, aka *direct revelation mechanisms*, \( \mathcal{G} = (\text{Agn}, \text{Str}, g) \) s.t:

- \( \text{Agn} \) is the finite set of agents of \( \mathcal{S} \);
- \( \text{Str} = \text{Prf} \), i.e., the set of strategy profiles is the set of preference profiles;
- \( g \) is the outcome function of the game: \( g : \text{Str} \rightarrow \text{Iss} \)

Agents play the game by declaring their preferences!
Truthful implementation

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Given a Direct Implementation Problem $(\mathcal{S}, \mathcal{G})$, find a direct revelation mechanism $G \in \mathcal{G}$ and a solution concept $S$ s.t.:

$$g(S(G, p)) = Sc(p)$$

If such a $G$ exists, then $Sc$ is truthfully $S$-implementable
Revelation in *Dominant Strategies*

A *dominant strategy equilibrium* of a strategic game \((G, \mathcal{P})\) with \(G = (\text{Agn, Str}, g)\), is a strategy profile \(s^* \in \text{Str}\) s.t. \(\forall i \in \text{Agn}, \text{ and } \forall s \in \text{Str}:\)

\[
g((s_{-i}, s_i)) \preceq_i g((s_{-i}, s_i^*))
\]

with \(\preceq_i\) being the i-th projection of \(\mathcal{P}\). A social choice function is said to be *DSE-implementable* if it is implementable w.r.t. dominant strategy equilibrium.

**Theorem (Revelation Principle).** Given an Implementation problem \((\mathcal{S}, \mathcal{G})\), if there exists a game form \(G \in \mathcal{G}\) DSE-implementing \(\mathcal{S}\), then there exists a direct revelation mechanism \(G^d\) s.t. \(\forall \mathcal{P} \in \text{Prf}\):

\[
\text{DSE}(G^d, \mathcal{P}) = \mathcal{P}
\]

that is, \(\mathcal{S}\) is *truthfully DSE-implementable*. 
Fact. If $\text{Sc}$ is truthfully DSE-implementable then it is incentive compatible, i.e.:

$$\forall p : DSE(G, p) = p \Rightarrow \forall i, \forall p, p' : Sc((p_{-i}, p'_i)) \preceq_i Sc((p_{-i}, p_i))$$

where $G$ is a direct revelation mechanism, and $\preceq_i$ is the $i^{th}$ projection of $p$.

- DSE implementation is strictly related to the problem of preference elicitation
No DSE implementation!

Truthful DSE-impl. 

- Revelation Principle
- DSE-impl.

Dict. 

- Gibb.-Satterth.
- Inc. Comp.
How is MD possible?

• No non-trivial incentive compatible social choice function!
• No DSE implementations of non-trivial social choice functions!

• Under what conditions can we prove the existence of non-trivial incentive compatible social choice functions become by designing mechanisms?
With Money
The virtues of money

1. Preference intensity can be measured and interpersonal comparisons become possible

2. The unit of measure of preference intensity is transferable. Payments become possible:

\[ u_i(a) = v_i(a) - p \]
Auctions

An auction is a Social Choice structure:

\[ G = \langle \text{Agn}, \text{Iss}, \text{Prf}, \text{Sc} \rangle \]

s.t:

- \text{Agn} is a finite set of agents such that \( 1 \leq |\text{Agn}|; \)
- \text{Iss} := \text{Agn} \times \mathbb{R};
- \text{Prf} is the set of all valuations of the auctioned item, i.e., \(|\text{Agn}|\)-tuples \( p = (w_i)_{i \in \text{Agn}} \) where each \( w_i' \in \mathbb{R}; \)
- \text{Sc} is a function \( \text{Sc} : \text{Prf} \rightarrow \text{Iss} \)

- Valuations determine total preorders on Iss
- \text{Sc} picks a winner and establishes a payment
What's the price?

1. Highest bidder wins, and no payment?
2. Highest bidder wins and pays the bid?

Vickrey Auction

Define $\mathcal{S}_c$ as follows. Let the winner be the agent $i$ with the highest declared valuation $w_i$ and let $i$ pay the second highest declared valuation $p = \max_{j \neq i} w_j$.

**Theorem.** Let $u(w_i)$ denote the utility of $i$ if $i$ bids $w_i$. For any profile of declared valuations $(w_1, \ldots, w_n)$, and valuation $w'_i$, it holds that $u(w'_i) \leq u(w_i)$.

- Possibility of an incentive compatible mechanism under a specific subclass of total orders!
- Mechanism design aims at the generalization of such possibility: between Vickrey and Gibbard-Satterthwaite.
Incentive compatible functions are implementable (with Money)! The possibility is proven by the existence of the Vickrey auction!
Without Money
A preference profile $\mathbf{p} = (\preceq_1, \ldots, \preceq_n)$ of total preorders on $\text{Iss}$ is single-peaked if there exists a total order $\preceq^*$ on $\text{Iss}$ s.t. $\forall i \in \text{Agn}$:

$$y \preceq_i x \& B(x, y, z) \implies z \prec_i y$$

where $B$ is the betweenness relation induced by $\preceq^*$. 
Examples

- Laws and policies (from LEFT to RIGHT)
- Locations (from FAR to CLOSE)
- Dimensions (from SMALL to BIG)

- NB: no money is involved in such decision!
- NB: no interpersonal comparison needed!
Possibility of Strategy-Proofness

Theorem (Pairwise majority). Take an Implementation problem \((\mathcal{S}, \mathcal{G})\) s.t. \(|\text{Agn}|\) is odd. The direct revelation mechanism \(G\) with outcome function \(g\) being \textit{pairwise majority voting} is an incentive compatible social choice rule under DSE.

Theorem (Median voter). Take an Implementation problem \((\mathcal{S}, \mathcal{G})\) s.t. \(|\text{Agn}|\) is odd. The mechanism \(G\) where: i) agents declare their peak; ii) the outcome function \(g\) selects the \textit{median voter}'s peak is an incentive compatible social choice rule under DSE.

- They are equivalent mechanisms
- Both select the unique \textit{Condorcet Winner}
Single-peaked preferences are sufficient to yield the possibility of incentive-compatible social choice functions (e.g., pairwise majority, median voter rule)
... Moral of the Story

- MD moves from the acknowledgment of two related impossibilities:
  - NO non-dictatorial incentive compatible social choice functions;
  - NO DSE-implementations of non-dictatorial functions.
- MD is developed by restricting the type of allowed preferences.